The Dynamics of Financial Policies and Group Decisions in Private Firms*

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Abstract

We model a private firm founded and run by a group of investors with heterogeneous capital contributions and risk preferences who decide on the firm’s financial policies and governance. Investors’ optimal claims resemble preferred stock with heterogeneous dividend caps, and common stock. Cashflow rights and control rights are separated and time-varying. The optimal investment policy is a time-varying weighted average of investors’ optimal policies and converges to the policy of the least (most) risk averse investor in booms (recessions). Optimal leverage is procyclical. The dynamic financial policies and diversity in equity claims resolve investors’ diverging preferences and inability to trade.

Keywords: group decisions, investment, payout, risk preference, governance

JEL: G32, G34, G35

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1 Introduction

The corporate finance literature has traditionally focussed on studying public corporations. In comparison, relatively little is known about private firms, even though they vastly outnumber their public counterparts. Private firms come under the form of legal entities such as private limited companies, limited liability companies (LLCs), sole proprietorships or various types of partnerships. Private companies share two key characteristics, the implications of which are underexplored: they are owned by a small group of investors and their shares are not traded on a public exchange. Shares in private firms therefore tend to be highly illiquid and investors are usually unable unilaterally to withdraw the capital or assets they contributed.

This paper develops a dynamic, open-horizon model of a private firm’s investment, financing and payout policies when these decisions are made by a small group of undiversified investors with heterogeneous risk preferences who cannot trade their claims on the firm. Our model focuses on Pareto optimal policies, which implies that group members maximize their joint, weighted life-time utility from the firm’s internally generated payouts. The members’ “utility weights” are endogenously determined at the startup by their participation constraint. A member’s utility weight increases with the capital she contributes at inception and the life-time utility she can achieve outside the private firm (i.e. as a sole proprietor). Together investors’ utility weights determine the firm’s internal governance.

The firm’s profits and dividends are subject to corporate taxes and personal taxes, respectively.¹ Dividends after taxes are investors’ only source of income as group members cannot trade their claims on the business. Except for these restrictions, group members otherwise operate in a frictionless environment where there are no synergies from combining assets (e.g. constant returns to scale), assets are perfectly divisible, and individuals

¹Some types of private firms avoid corporate taxes. For example, profits and losses of partnerships are typically not taxed at the corporate level, but pass directly through the owners and are taxed at the personal level only.
have the same investment opportunity set as groups. Investment in risky assets and payouts are made in continuous time. We assume that investors must agree on one single investment policy and capital structure for the firm, but investors can agree to tailor payouts to the preferences of individual group members. This flexibility in designing investors’ claims turns out to be important given that investors have different preferences and are unable to trade their claims on the private firm.

What investment policy do investors agree on, given that they have different preferences, and therefore would like to adopt different policies? The literature offers two competing hypotheses for group decisions. The group shift hypothesis (e.g. Moscovici and Zavalloni (1969); Kerr (1992)) suggests that group decisions shift towards the one of the dominant person in a team. As that person typically holds very pronounced preferences, a team eventually gravitates towards extremes. Consequently, teams make more polarized decisions than individuals do. In contrast, an alternative hypothesis predicts that extreme preferences in a group are averaged out and teams eventually make less extreme decisions than individuals do. Although this kind of diversification hypothesis appears to enjoy strong empirical support (see Bär, Kempf and Ruenzi (2011)), it struggles to explain the puzzling observation that the average group is more (less) risk averse than the average individual in high (low) risk situations (Shupp and Williams (2008)).

We show that the optimal investment policy for the group is a weighted average of each individual investor’s optimal policy, consistent with the diversification hypothesis. However, the weights vary with the firm’s net worth. As net worth increases (decreases) in good (bad) times, increasingly more weight is put on the least (most) risk averse investor, causing the group’s coefficient of risk aversion – defined in terms of the group’s indirect utility function – to converge to the lowest (highest) coefficient of risk aversion in the group, which is consistent with the group shift hypothesis. Consequently, the fraction of the firm’s net worth invested in risky assets is not constant, but increases (decreases) in good (bad) times. Leverage is procyclical: the firm’s net debt ratio (NDR) rises when the firm is doing well, and falls in bad times. Startups with little net worth optimally
hold a net cash position (i.e. negative debt). As a firm and its net worth grow, it starts borrowing using a line of credit and builds up a net debt position. Firms dynamically delever (lever up) towards the NDR of the most (least) risk averse investor as net worth shrinks (grows). Only when all investors have the same coefficient of risk aversion does the firm maintain a constant NDR. By dynamically rebalancing assets and liabilities in response to income shocks, debt is kept safe at all times as investors avoid default due to risk aversion and their reliance on the firm for future consumption. As far as we are aware, these results are original.

Under the firm’s optimal policies, each investor receives a claim that specifies the investor’s payout as a function of the firm’s net worth. The dynamic payout and investment policies are time-consistent because no learning takes place along the way, and investors’ exogenous risk preferences and endogenous utility weights do not change over time.

We find that the payout and claim value of the least (most) risk averse investor is a convex (concave) function of the firm’s net worth. For investors with intermediate levels of risk aversion, the payout and claim value are S-shaped in net worth (i.e. convex for low levels of net worth, and concave for high levels). Payouts and claim values converge to fixed caps as net worth increases, except for the least risk averse investor whose payout and claim become arbitrarily large as net worth goes to infinity. We show that the claim on the firm’s assets of the least risk averse investor is similar to a common equity contract. The claims of investors with higher levels of risk aversion resemble preferred stock with more risk averse investors having a higher seniority but lower maximum dividend. Only when all investors have the same coefficient of risk aversion do we obtain payouts and claims that are linear in the firm’s assets and net worth, reflecting equity contracts in which each investor gets a fixed proportion of the firm’s total payouts and owns a constant fraction of the firm’s net worth. Our model shows that an investor’s payout and claim value depend not only on her own risk preferences, but also on the risk preferences of her co-investors. Investors that are relatively less risk averse opt for riskier claims with higher upside potential but more downside risk.
There exists a unique set of utility weights for which all investors are indifferent between operating independently as a sole proprietor or as part of a group. Any other combination of weights makes at least one investor worse off, violating the participation constraint. Under the chosen utility weights and the optimal policies, investors within the group receive the same payouts and accumulate the same wealth as if they were operating on their own. Therefore, it does not matter whether and when they join the group. This is an example of Coase theorem at work. Given there are no frictions nor operational synergies from combining assets (assets are perfectly divisible and subject to constant returns to scale), it does not matter whether or not an investor is part of a group. All investors have access to the same investment opportunity set and are subject to the same economics shocks and taxation laws. There are no diversification benefits or synergies from operating in a group compared to operating alone. Of course, within a group, investors “share risk” in the sense that less risk averse investors absorb most losses in bad times and capture most gains in good times. The firm’s financial structure is therefore optimal not only from a taxation viewpoint (higher corporate taxes generate a higher NDR), but the optimal mix of net debt, equity and preferred equity claims also efficiently share risk among investors.

Each investor’s utility weight is increasing in her capital contribution at inception. The relation is non-linear except when all investors have log utility, in which case an investor’s relative weight is proportional to her capital contribution. The utility weights not only influence investors’ payout but also their relative control over the firm. Less risk averse investors (e.g. the entrepreneur) exert more influence over corporate decision making when the firm is doing well, whereas more risk averse investors (e.g. the venture capitalists) assume more control when the firm is performing poorly. These results echo some of the findings of the incomplete contracting literature (e.g. Aghion and Bolton (1992)) and they are also consistent with the empirical findings of Kaplan and Strömberg (2003) on venture capital contracts. Our model generates these results without the usual

\[^2\]The utility weights are unique only in relative terms, not in absolute terms. In other words the weight of one of the investors serves as a numeraire.
frictions such as asymmetric information and agency costs. Our results follow from the fact that agents have heterogeneous preferences and are unable to trade their claims on the firm.

Our model endogenously generates contracts resembling preferred stock for which cash flow rights may differ from control rights. Gilson and Schizer (2003) argue that preferred stock facilitates the separation of control and cash flow rights. Venture capitalists typically have more control rights than cash flow rights (for example, they might claim more than half of the board seats but only one-quarter of the profits). The conflicts among agents arising from the differences in risk aversion are resolved via dynamic rebalancing of cashflow rights and control rights. Therefore our contracts are dynamically optimal.

Investment in risky assets and the firm’s NDR increase with the corporate taxation rate but are independent of personal taxation, just like in a sole proprietorship. Payouts are independent of personal taxation because the governance structure (i.e. investors’ utility weights) implemented at startup ensures that no investor is worse off compared to what she could achieve in a sole proprietorship. Unexpected changes in the tax rates after startup lead, however, to wealth transfers among investors.

Our paper relates to two strands of papers: a literature on dynamic group decision making, and a recent literature on continuous-time models of entrepreneurial finance. Dynamic models of group decision making in corporate finance are very rare.\(^3\) A notable exception is Garlappi, Giammarino and Lazrak (2017) who study a dynamic corporate investment problem by a group of agents holding heterogeneous beliefs and adopting a utilitarian aggregation mechanism. They show that group decisions are dynamically inconsistent due to learning and that this may lead to underinvestment. Garlappi, Gi-

\(^3\)Most existing models on group decisions are static in nature and build on a seminal literature on partnerships and syndicates that studies how risk should be shared between partners (see e.g. Wilson (1968), Eliashberg and Winkler (1981), Pratt and Zeckhauser (1989)). More recently, Mazzocco (2004) examines the savings behavior of couples with homogenous HARA preferences and individual stochastic incomes. Hara, Huang and Kuzmics (2007) show within a static setting that heterogeneity in consumers’ risk attitudes generates optimal sharing rules that are concave, convex, or initially convex and eventually concave, depending on investors’ risk preferences.
ammarrino and Lazrak (2021) study a real option model where the decisions to invest in and abandon a project are made sequentially by a group of agents with heterogeneous beliefs who make decisions based on majority voting. Voting leads to inefficient underinvestment when group members’ beliefs are polarized. Ebert, Wei and Zhou (2020) model a canonical real option investment problem under weighted discounting. Weighted discount functions may describe the discounting behavior of groups. They find that greater group diversity leads to delayed investment. Some papers examine group decision within a principal-agent setting. For example, Grenadier, Malenko and Malenko (2016) model a real options investment decision in an organization where an uninformed principal makes a timing decision interacting with an informed but biased agent. They examine the optimality of centralized decision making (i.e. where the principal retains authority and communicates with the agent) versus delegation. These papers do not consider optimal capital structure, payout structure, and internal governance, and all agents have identical (risk neutral) risk preferences.


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4There is also a growing literature on decisions and voting behavior by corporate boards (see e.g. Levit and Malenko (2011) and Malenko (2014), Donaldson, Malenko and Piacentino (2020) among others).

5A large body of dynamic asset pricing papers studies the effect of heterogeneous risk preferences on prices in a market equilibrium framework. For example, Dumas (1989), Wang (1996), Cvitanić et al. (2012), and Bhamra and Uppal (2014) consider a pure exchange economy where agents can trade their share in the exogenous, aggregate endowment of the consumption good and have access to a risk-free security. In our setting the aggregate cash-flows are endogenously generated with group members exerting time-varying control on the investment decision. Although group members cannot trade their claims, non-linearities and heterogeneity in members’ claims allow for time-variation in relative consumption entitlements. Our model easily encompasses \( n \) different agents, whereas many asset pricing models are specialized to two classes of consumers.
a risk-averse Limited Partner (LP) and investigate whether the performance of private equity investments is sufficient to compensate LPs for risk, long-term illiquidity, management, and incentive fees charged by the general partner (GP). Gornall and Strebulaev (2020) derive the fair value of VC-backed companies and of each class of share they issue, taking into account the intricacies of contractual cash flow terms. Whereas existing papers focus on the decisions of a single entrepreneur, and take contracts as exogenously given, our paper considers a group of investors, endogenizes their optimal claims and the firm’s governance structure.\(^6\) Our paper does not consider nondiversifiable risks. Although investors in our model cannot trade their ownership stakes in the firm, the optimal design of the investors claim completes the market, and synthetically replicates a Merton (1969) environment for each investor.

The previously mentioned entrepreneurial finance papers have branched out from a growing literature in corporate finance that jointly models a firm’s investment, payout, borrowing and closure decisions in a dynamic framework. Recent continuous-time papers in this strand include Gryglewicz (2011), Bolton, Chen and Wang (2011), Décamps, Gryglewicz, Morelec and Villeneuve (2017), and Lambrecht and Myers (2017), among others. In these papers the firm’s financial policies are set by a single decision maker.

\section{The Model Setup}

Consider a group of \(n\) investors who consider starting up a private firm at time \(t = 0\). Each investor can contribute an amount of capital \(W_{i0}\), such that the firm’s total net worth upon inception equals \(W_t = \sum_{i=1}^{n} W_{i0}\). The firm thereafter invests an amount \(A_t\) in risky assets that generate an after-tax return given by the diffusion process:

\[
\frac{dA_t}{A_t} = \mu'(1 - \tau_c)dt + \sigma'(1 - \tau_c)dB_t \equiv \mu dt + \sigma dB_t
\]  \hfill (1)

\(^6\)Heinkel and Zechner (1990) analyze the optimal mix of debt, common equity, and preferred equity. They provide a rationale for the use of preferred equity in the presence of asymmetric information and taxes.
where $B_t$ is a Brownian motion and $\tau_c$ is the corporate tax rate. The drift and volatility of the process are $\mu$ and $\sigma$ respectively. We assume that assets are perfectly divisible, and subject to constant returns to scale. Without loss of generality, we assume that there is only one risky asset. Our results hold for a framework with multiple risky assets, provided that all investors have access to the same investment opportunity set. In that case the risky asset can always be thought of as a composite asset (e.g. a portfolio of different projects).\footnote{\cite{Merton1971} shows that when asset prices are lognormally distributed all investors hold a combination of the risk-free asset and a mutual fund. The proportions of each asset held by the mutual fund only depend on the price distribution parameters and are independent of individual preferences and wealth distribution.}

The firm finances its assets with a line of credit, $D_t$, and equity, $W_t$, i.e $A_t = W_t + D_t$. $D_t$ is the amount of net debt. If $D_t$ is negative, then the firm has a net cash position. The firm can continuously roll over its net debt position. In what follows we show that the firm never defaults under the optimal policies and that debt is risk free. The firm can therefore borrow and save at the risk free rate, and the firm’s instantaneous after-tax cost (return) of debt (cash) is $r$, i.e.

$$\frac{dD_t}{D_t} = r'(1 - \tau_c)dt \equiv r dt \quad (2)$$

We define $\xi \equiv \frac{\mu - r}{\sigma}$ as the Sharpe ratio associated with the firm’s risky assets.

At each instant $t$ the group members receive payout at a rate $c_{it}$ ($i = 1, ..., n$), and the firm invests an amount $A_t$ in risky assets, given the firm’s net worth $W_t$. The net worth process is therefore:

$$dW_t = dA_t - dD_t - \left(\sum_{i=1}^{n} c_{it}\right)dt = \left[(\mu - r)A_t + rW_t - \sum_{i=1}^{n} c_{it}\right]dt + \sigma A_t dB_t \quad (3)$$

Define $\omega_t \equiv \frac{A_t}{W_t}$ as the fraction of the firm’s net worth invested in risky assets. If $\omega_t > 1$ then the firm invests all its net worth $W_t$ in risky assets and borrows an amount $(\omega_t - 1)W_t$
that is also invested in risky assets. The process for \( W_t \) is now

\[
dW_t = \left[ (\omega_t(\mu - r) + r) W_t - \sum_{i=1}^{n} c_{it} \right] dt + \sigma \omega_t W_t dB_t
\]  

(4)

All group members have a power utility function and therefore constant relative risk aversion. Members can, however, have a different coefficient of risk aversion \( \gamma_i \), i.e. investor \( i \)'s utility function is given by

\[
u_i(c_{it}) = c_{it}^{1-\gamma_i} - \frac{1}{1-\gamma_i}
\]

where \( \lim_{\gamma_i \to 1} u_i(c_{it}) = \ln(c_{it}) \). All investors are strictly risk averse (\( \gamma_i > 0 \)) and have a subjective discount rate \( \rho \). Without loss of generality, we assume \( \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n \).

Each investor wants to maximize the expected life-time utility from her payouts. They only join the group if their expected life-time utility from investing in the private firm is at least as high as what they can achieve outside the firm. Given that assets are perfectly divisible and subject to constant returns to scale, it is possible for each investor to run a firm on her own as a sole proprietorship. Assuming that investor \( i \) is endowed with wealth \( W_{i0} \), the life-time utility she can generate from the sole proprietorship is given by the familiar Merton (1969) solution.

\[
J_i(W_{i0}) = \max_{\{c_{it}^S, \omega_{it}^S\}} \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} u_i((1-\tau_p)c_{is}) ds \right] = a_{it}^\gamma \frac{W_{i0}^{1-\gamma_i}}{1-\gamma_i} - \frac{1}{\rho(1-\gamma_i)}
\]

(5)

with \( \omega_{it}^S = \xi/(\sigma \gamma_i) \) and \( c_{it}^S = W_{it}/a_i \). \( \tau_p \) is the personal tax rate and \( a_i \) is defined in Equation (13) below. The superscript \( S \) refers to the sole proprietorship case.

Instead of running a sole proprietorship, investor \( i \) can join a group with total net worth \( W_t \), in which case her life-time utility \( J_i(W_t) \) is:

\[
J_i(W_t) = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} u_i((1-\tau_p)c_{is}) ds \right]
\]

(6)

\footnotetext{In the Internet Appendix C we present the results for investors with exponential utility \( u_i(c_{it}) = -\exp(-\eta_i c_{it})/\eta_i \).}
It follows that investor $i$’s participation constraint at startup ($t = 0$) is satisfied if:

$$J_i(W_0) \geq J_i(W_{i0}) \quad (7)$$

Investors in the private firm must, however, jointly agree on the payout policies $c_{it}$ ($i = 1, \ldots, n$) and the investment policy $\omega_t$. We focus on Pareto efficient payout policies among the $n$ members. That is, we identify policies that are not dominated by any other contract when considered from the perspective of all investors as individuals. We know (see Duffie (2001), Chapter 10) that a necessary and sufficient condition for $(c_{1t}, \ldots, c_{nt})$ to be a Pareto efficient allocation is that there exist non-negative weights $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $(c_{1t}, \ldots, c_{nt})$ and $\omega_t$ are the solution to the following optimization problem:

$$J(W_t) = \max_{\{c_{1t}, \ldots, c_{nt}, \omega_t\}} \mathbb{E}_t \left[ \sum_{i=1}^n \int_t^\infty e^{-\rho(s-t)} \lambda_i u_i((1 - \tau_p) c_{is}) ds \right] \equiv \sum_{i=1}^n \lambda_i J_i(W_t) \quad (8)$$

subject to the intertemporal budget constraint (3), the participation constraint (7) and the transversality condition $\lim_{t \to \infty} \mathbb{E}_t [e^{-\rho t} J(W_t)] = 0$. Objective function (8) is known as the weighted-sum-of-utilities welfare function, a generalization of the classical utilitarian welfare function. Note that group members derive utility from dividends $c_{it}$ only (i.e. they do not sell their claim to generate income). This completes the formulation of the optimization problem.

3 Optimal Financial Policies

The below proposition presents the solution to the optimization problem (8).
Proposition 1 The private firm’s optimal investment ($\omega_i^*$) and payout ($c_{it}^*$) policies are:

$$\omega_i^* = \left( \frac{A_i}{W_i} \right)^* = \frac{\xi W'(z_t)}{\sigma W(z_t)} = \sum_{i=1}^{n} \frac{\omega_i(z_t)}{\sigma\gamma_i} = \sum_{i=1}^{n} \frac{c_{it}^*}{\omega_i^*\sigma\gamma_i}$$  \hspace{1cm} (9)$$

$$c_{it}^* = (1 - \tau_p)\frac{1}{\gamma_i} - \frac{1}{\lambda_i} e^{z_t} \equiv \omega_i^* \frac{W_i}{a_i} \text{ for } i = 1, \ldots, n$$  \hspace{1cm} (10)$$

where $z$ is the solution to

$$W(z_t) = \sum_{i=1}^{n} G_i e^{\frac{z_t}{\gamma_i}} \text{ and where}$$  \hspace{1cm} (11)$$

$$\omega_i(z_t) = \frac{G_i e^{\frac{z_t}{\gamma_i}}}{\sum_{i=1}^{n} G_i e^{\frac{z_t}{\gamma_i}}} \text{ and } \sum_{i=1}^{n} \omega_i(z_t) = 1$$  \hspace{1cm} (12)$$

$$G_i = \frac{\gamma_i^2 (1 - \tau_p)\frac{1}{\gamma_i} - \frac{1}{\gamma_i} \lambda_i^\frac{1}{\gamma_i}}{r\gamma_i^2 + (\rho + \frac{\xi^2}{2} - r)\gamma_i - \frac{\xi^2}{2}} \equiv \hat{a}_i(1 - \tau_p)\frac{1}{\gamma_i} - \frac{1}{\lambda_i} \equiv a_i\lambda_i^\frac{1}{\gamma_i}$$  \hspace{1cm} (13)$$

The group’s joint weighted life-time utility, $J(W(z_t))$, and utility weights, $\lambda_i$, are given by:

$$J(W(z_t)) = \sum_{i=1}^{n} \lambda_i J_i(z_t) = \sum_{i=1}^{n} \lambda_i \left[ \frac{a_i}{1 - \gamma_i} (\lambda_i e^{z_t})\frac{1}{\gamma_i} - \frac{1}{\rho(1 - \gamma_i)} \right]$$  \hspace{1cm} (14)$$

$$\lambda_i = \left( \frac{W_i}{a_i} \right)^{\gamma_i} e^{-z_0} = \frac{e^{-z_0}}{u_i'(1 - \tau_p)\sigma_{i0}} \text{ with } \frac{u_i'(1 - \tau_p)c_{i0}^*}{dJ_i(W_{i0})/dW_{i0}} = 1 \text{ for } i = 1, \ldots, n$$  \hspace{1cm} (15)$$

The investment and payout policies, and the firm’s weighted claim value are not explicitly expressed as a function of its total net worth $W_t$ but as a function of an auxiliary variable $z_t$. Net worth is, however, a continuous, monotonically increasing function of $z$ as expressed by Equation (11). As $z_t$ ranges from $-\infty$ to $+\infty$, $W_t$ varies from 0 to $+\infty$, i.e. $\lim_{z_t \to -\infty} W(z_t) = 0$ and $\lim_{z_t \to +\infty} W(z_t) = +\infty$. The firm never defaults because zero net worth is never reached.

For every value of $z_t$, there is a unique corresponding value for $W_t$. For expositional purposes it will be easier to express our result in terms of $z_t$, rather than $W_t$. $z_t$ can be
interpreted as some kind of “normalized” wealth variable.

3.1 Investment policy

From Merton (1969, 1971) we know that the optimal investment and payout policies of a single investor $i$ with wealth $W_t$ are, respectively, given by $\omega_i^S = \xi/(\sigma \gamma_i)$ and $c_{it}^S = W_t/\hat{a}_i$. We find that the optimal investment policy $\omega_t^*$ of a coalition of investors is a weighted average of the optimal investment policies $\omega_i^S$ of the individual investors. The weight $\omega_{it}$ reflects member $i$’s influence or “control” over the investment policy. We can rewrite $\omega_{it}$ as

$$\omega_{it} = \frac{c_{it}^*}{W_t/\hat{a}_i} = \frac{c_{it}^*}{c_{it}^S}$$

The optimal control weights $\omega_{it}$ (not to be confused with the “utility” weights $\lambda_i$) are given by the ratio $c_{it}^*/c_{it}^S$ of the individuals’ payout $c_{it}^*$ and her optimal payout $c_{it}^S$ if she ran the whole firm alone. Importantly, the control weights $\omega_{it}$ are time-varying as they depend on the firm’s net worth $W_t$ through the auxiliary variable $z_t$.

**Corollary 1** The firm’s risky asset to net worth ratio, $\omega_t^*$, increases in its net worth, i.e. $\frac{\partial \omega_t^*}{\partial W_t} \geq 0$

The corollary implies that the firm invests more aggressively in risky assets in good times (when its net worth has grown), and reduces its weight in risky assets during bad times.\(^9\) Consider the $n$-investor case where agent 1 ($n$) is least (most) risk averse (i.e. $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$). In that case,

$$\lim_{z_t \to +\infty} \omega^*(z_t) = \frac{\xi}{\sigma \gamma_1} \quad \text{and} \quad \lim_{z_t \to -\infty} \omega^*(z_t) = \frac{\xi}{\sigma \gamma_n}$$

As the firm accumulates (loses) more wealth, its investment policy converges towards the one of the least (most) risk averse partner. $\omega^*(z_t)$ is an S-shaped function situated between

\(^9\)The proofs of the corollaries are in the Internet Appendix A.
the asymptotes $\frac{\xi}{\sigma \gamma_1}$ and $\frac{\xi}{\sigma \gamma_n}$. $\omega^*(z_t)$ is constant for the knife-edge case of homogenous risk preferences.

For the special case where all members have the same degree of risk aversion $\gamma$, the group’s investment policy is independent of time and given by the standard Merton (1969) ratio: $\omega = \xi/(\sigma \gamma)$.

### 3.2 Debt policy

The private firm’s net debt level is given by $D_t = (\omega^*_t - 1)W_t$. If $D_t < 0$ then the firm’s cash-holdings exceed its debt, so that the firm has a net cash surplus (or negative debt). The below corollary states the firm’s net debt ratio under the optimal investment and payout policy. With negative debt, the traditional net debt ratio (NDR) is hard to interpret. Lambrecht and Pawlina (2013) provide an economic rationale for using the following NDR defined over the interval $[-1, 1]$.

**Proposition 2** Define the firm’s net debt ratio as $NDR_t \equiv \frac{D_t}{W_t + \Psi D_t}$ where $\Psi = 1$ if $D_t \geq 0$ and $\Psi = 0$ if $D_t < 0$. The NDR is time-varying, procyclical, and given by

$$-1 \leq \frac{\xi}{\sigma \gamma_1} - 1 \frac{1}{1 + (\frac{\xi}{\sigma \gamma_1} - 1)\Psi} < NDR_t = \frac{\omega^*_t - 1}{1 + (\omega^*_t - 1)\Psi} < \frac{\xi}{\sigma \gamma_1} - 1 \frac{1}{1 + (\frac{\xi}{\sigma \gamma_1} - 1)\Psi} \leq 1 \quad (16)$$

The firm’s leverage is procyclical and increases in $W_t$. For low levels of net worth (e.g. at startup) and assuming some agents are sufficiently risk averse, the firm initially keeps a net cash balance and only invests a fraction of its net worth in risky assets (i.e. $\omega^*_t < 1$). As the firm accumulates retained earnings, the optimal NDR rises and converges towards the optimal NDR of the least risk averse investor, which involves net borrowing if some agents have a sufficiently low coefficient of risk aversion. In bad times, as net worth shrinks, the firm rebalances by selling or shutting down risky assets, and using the proceeds to pay down its line of credit. As $W_t \to 0$ (or $z_t \to -\infty$), the NDR converges towards the
most risk-averse investor’s optimal NDR. The procyclical leverage together with the time-varying investment policy implies that in good times, the firm leverages up and invests more aggressively in risky assets, while in bad times, it reduces leverage and adopts a more conservative investment strategy. Such dynamic rebalancing in response to shocks allows the firm to keep debt safe all the time. In other words, safe debt arises endogenously from our model because of the firm’s ability to rebalance assets and liabilities.

As investor \( n \) becomes infinitely risk averse \( (\gamma_n \to +\infty) \), the lower bound for the NDR converges to -1, which represents an all equity-financed firm that invests in the safe asset (cash) only. As investor 1 moves towards risk neutrality \( (\gamma_1 \to 0) \), the upper bound for the NDR converges to +1, which represents a near 100% debt-financed firm that invests in risky assets only. Therefore, if the most and least risk averse agents have highly divergent risk preferences, then the intertemporal variation in the firm’s NDR across the business cycle could be large.

### 3.3 Payout policy

The payout (in dollars) to each partner is increasing in the firm’s net worth. Unlike Merton (1969), where payout is linear in wealth, payout is non-linear in total net worth if investors have different levels of risk aversion. Since \( c_{it}^* = \omega_{it} W_t / \hat{a}_i = (1-\tau_p) \frac{1}{\gamma_i} \omega_{it} W_t / a_i \) (see Equation (10)), the non-linearity is driven by the time-varying control weight \( \omega_{it} \).

For the \( n \)-investor case one can prove the following proposition:

**Proposition 3** Payout is a convex (concave) increasing function of the firm’s total net worth \( W_t \) for the least (most) risk averse investor. For investors with an intermediate level of risk aversion, there exists a critical wealth level \( W_i^* \) \( (i = 2, 3, \ldots, n - 1) \) such that payout is convex in wealth for low levels of wealth (i.e. \( W_t \leq W_i^* \)) and concave in wealth for high levels of wealth (i.e. \( W_t \geq W_i^* \)). Furthermore, \( \lim_{z_t \to -\infty} c_{it}^* = 0 \) for \( i = 1, \ldots, n \), and

\[
\lim_{z_t \to +\infty} \frac{\partial c_{it}^*}{\partial W_t} = \frac{1}{\hat{a}_i} \quad \text{and} \quad \lim_{z_t \to +\infty} \frac{\partial c_{it}^*}{\partial W_t} = 0 \quad \text{for} \quad j = 2, 3, \ldots, n
\]
Payout is linear in net worth if investors have homogeneous risk preferences:

\[ \frac{\partial^2 c^*_{it}}{\partial W_t^2} = 0 \text{ and } c^*_{it} = \frac{\lambda^\frac{1}{\gamma_i} W_t}{\sum_{j=1}^{n} \lambda^\frac{1}{\gamma_j}} \text{ for } i=1,\ldots,n \Leftrightarrow \gamma_1 = \ldots = \gamma_n = \gamma \quad (17) \]

It is Pareto optimal for the least (most) risk averse agent to have a convex (concave) payout function. Since all payouts are zero for \( W_t = 0 \), it follows that, among all investors, the most risk averse investor is paid the most for very low levels of net worth, whereas the least risk-averse investor gets paid the most for very high levels of net worth. The least risk averse investor is the residual claimant who gets the upside potential in good times, but is heavily exposed in bad times. All other risk averse investors are less exposed in downturns, but their payouts are capped as the firm grows and accumulates net worth. The diversity in payouts reconciles group members’ heterogeneous risk preferences, allows for efficient risk sharing, and substitutes for the lack of flexibility in the firm’s ownership (group members cannot trade their claims). Each investor’s payout is non-linear in total net worth, except if all investors have the same coefficient of risk aversion. These findings are also reflected in investors’ relative cash flow rights:

\[ \frac{c^*_{jt}}{c^*_{it}} = \frac{\lambda^\frac{1}{\gamma_j} e^z}{\lambda^\frac{1}{\gamma_i}} (1 - \tau_p)^{\frac{1}{\gamma_j} - \frac{1}{\gamma_i}} \quad (18) \]

Equations (17) and (18) lead to the following corollary.

**Corollary 2** Each investor receives a constant fraction of total payout if and only if all investors have the same risk preferences. An investor’s payout is strictly proportional to her utility weight \( \lambda_i \) if and only if all investors have logarithmic utility.

Consider next the firm’s payout yield \( c^*_t/W_t \). It follows from Proposition 1 that:

\[ \frac{c^*_t}{W_t} = \sum_{i=1}^{n} \frac{c^*_{it}}{W_t} = \sum_{i=1}^{n} \frac{\omega_{it}}{d_t} \quad (19) \]
The payout yield or average propensity to consume, \( c_t^*/W_t \), is time-varying, except if all investors have the same coefficient of risk aversion \( \gamma \) (and therefore \( a_i = a \) for all \( i \)):

\[
\frac{c_t^*}{W_t} = \frac{\sum_{i=1}^{n} (1 - \tau_p)^{\frac{1}{\gamma} - 1} \lambda_i^\frac{1}{\gamma} e^{\frac{z_t}{\gamma}}}{\sum_{i=1}^{n} a \lambda_i^\frac{1}{\gamma} e^{\frac{z_t}{\gamma}}} = \frac{1}{\hat{a}} = \frac{\rho}{\gamma} - \frac{r(1 - \gamma)}{\gamma} - \frac{\xi^2 (1 - \gamma)}{2\gamma^2} \quad \text{if } \gamma_i = \gamma \quad \text{for all } i \quad (20)
\]

This is the same average propensity to consume as in Merton (1969, Eq. (40)).

### 3.4 Investors’ equity claims

From Equation (15), which pins down the utility weights \( \lambda_i \), and Equation (10) for \( c_t^* \), it follows that the monetary certainty equivalent value \( W_{it} \) of each investor \( i \)’s claim is

\[
W_{it} = a_i \lambda_i^\frac{1}{\gamma} e^{\frac{z_t}{\gamma}} = a_i (1 - \tau_p)^{\frac{1}{\gamma} - 1} c_t^* = \hat{a}_i c_t^* = \omega_t W_t \quad \text{with } \sum_{i=1}^{n} W_{it} = W_t \quad (21)
\]

Using Equation (5) for \( I_i(W_{it}) \) and Equation (14) for \( J_i(W_t) \) it follows that:

\[
J_i(W_t) = I_i(W_{it}) \iff W_{it} = a_i \lambda_i^\frac{1}{\gamma} e^{\frac{z_t}{\gamma}} \quad (22)
\]

Hence, it follows from (21) and (22) that investors’ participation constraint is binding at all times, even though their utility weights remain fixed over time. The certainty equivalent wealth, \( W_{it} \), corresponds to the dollar net worth stake in the private firm for which investor \( i \) is indifferent between being a member of the group or running a sole proprietorship. By plowing an amount \( W_{it} \) in a sole proprietorship, investor \( i \) can achieve the same life-time utility as in the group. This amount \( W_{it} \) may differ from \( W_{i0} \), the amount invested at time 0. \( W_{it} \) will be larger (smaller) than \( W_{i0} \) if the firm has increased (decreased) its total net worth \( W_t \). \( W_{it} \) depends on the investor’s coefficient of risk aversion \( \gamma_i \). From Equation (21) it follows that investors’ control weights \( \omega_{it} \) also correspond to their implicit ownership share in the firm’s net worth. Investors’ implicit ownership share is time-varying. Note, however, that investors’ relative control rights may...
differ from investors’ relative cashflow rights since $c^*_i/c^*_j = (\omega_i\hat{a}_i)/(\omega_j\hat{a}_j) \neq \omega_i/\omega_j$.

The following corollary describes investors’ certainty equivalent value as a function of the firm’s net worth. Since this value is a constant multiple of payout (i.e., $W_i = \hat{a}_i c^*_i(W(z_t))$) the corollary mirrors the results we obtained in Proposition 3 for investors’ payout policies $c^*_i$.

**Corollary 3** The claim value of the most (least) risk averse investor is a concave (convex) function of the firm’s total net worth $W_t$. The claim of investors with intermediate levels of risk aversion is S-shaped in net worth. Moreover, $\lim_{z_t \to -\infty} W_it = 0$ for $i = 1, ..., n$, and

$$
\lim_{z_t \to +\infty} \frac{\partial W_{it}}{\partial W_t} = 1 \quad \text{and} \quad \lim_{z_t \to +\infty} \frac{\partial W_{it}}{\partial W_t} = 0 \quad \text{for} \quad i = 2, ..., n \quad (23)
$$

$$
\lim_{z_t \to -\infty} \frac{\partial W_{nt}}{\partial W_t} = 1 \quad \text{and} \quad \lim_{z_t \to -\infty} \frac{\partial W_{it}}{\partial W_t} = 0 \quad \text{for} \quad i = 1, ..., n - 1 \quad (24)
$$

Finally, $\exists W^*_i$ such that $\frac{\partial^2 W_{it}}{\partial W_t^2} \geq (\leq) 0 \iff W_t \leq (\geq) W^*_i \quad \text{for} \quad i = 2, ..., n - 1$.

As the firm becomes very wealthy, an additional dollar of net worth is almost entirely to the benefit of the least risk averse investor (i.e., $\lim_{z_t \to +\infty} \frac{\partial W_{1t}}{\partial W_t} = 1$), whereas the other investors’ claims converge to a fixed value. Conversely, for a firm with little or no wealth, the marginal dollar accrues almost entirely to the most risk averse investor (i.e., $\lim_{z_t \to -\infty} \frac{\partial W_{nt}}{\partial W_t} = 1$).

The strictly convex claim of the least risk averse investor ($i = 1$) is unbounded and resembles a common stock contract. The claims of investors 2 to $n-1$ are convex (concave) for low (high) values of net worth, with each claim converging towards a maximum value and resembling a preferred stock contract. Preferred stocks are widely used in venture capital financing. Preferred stock contracts are senior to common stock in that dividends may not be paid to common stock, unless the dividend is paid on all preferred stock. The

---

10 The corollary can be reformulated to describe claim values as a function of the firm’s holding in risky assets $A_t$. The results are very similar.
dividend rate on preferred stock is usually fixed at the time of issue. The preferred stock is junior to the debt claim $D_t$. Unlike debt contracts, the firm is not in default when it suspends dividends to preferred stockholders. In simple terms, one could view preferred stock as a debt instrument when dividends are not in arrears, and as an equity instrument when dividends are getting significantly in arrears. This explains why preferred stock values can be convex and concave for, respectively, low and high values of net worth $W_t$.

The claim of the most risk averse investor $(i = n)$ is strictly concave in $W_t$, and corresponds to the most senior preferred stock contract. For the lowest levels of net worth, an extra dollar for the firm accrues almost entirely to the most risk averse investor. As the firm accumulates more net worth, more junior preferred stock holders start sharing in the payouts. The claims of these other investors resemble preferred stock contracts for which the level of seniority increases in the degree of risk aversion, and for which the face value of the contract (achieved as $W_t \to +\infty$) decreases in the degree of risk aversion. The maximum value of each preferred stock contract is determined by the maximum payout rate $c_{it}$ as $W_t \to +\infty$. Our contracts resemble the ones in ventures where the inside entrepreneur captures most of the upside in return for bearing the downside risk, whereas outside investors are protected on the downside but capture less of the upside.

The variation in the shapes of the claim values is entirely driven by differences in investors’ coefficient of risk aversion, as illustrated by the following corollary:

**Corollary 4** If all investors have the same coefficient of risk aversion $(\gamma_1 = \gamma_2 = \ldots = \gamma_n)$, then the investors’ claim values are linear in the firm’s total net worth, i.e.

$$W_{it} = \frac{\lambda_i^\frac{1}{\gamma_i} W_t}{\sum_{i=1}^n \lambda_i^\frac{1}{\gamma_i}} \quad \text{for} \quad i = 1, 2, \ldots, n$$

(25)

If all investors have the same coefficient of risk aversion, then they share profits according to fixed proportions. This gives rise to common stock contracts that are linear in net worth. If all investors have log utility ($\gamma_i = 1$ for $i = 1, \ldots, n$) and the utility weights sum
to one, then each investor’s certainty equivalent claim value is proportional to her utility weight \( \lambda_i \) (i.e. \( W_t = \lambda_i W_t \)) and utility weights can be interpreted as ownership shares.

### 3.5 Utility weights, control weights and governance structure

From Equation (15) we know that the utility weights are given by:

\[
\lambda_i = \left( \frac{W_{it}}{a_i} \right)^{\gamma_i} e^{-z_0} = \frac{J'(W_0)}{u'_i((1 - \tau_p)c_{i0})} \quad \text{with} \quad \frac{u'_i((1 - \tau_p)c_{i0})}{\frac{dW_t}{dW_{i0}}} = 1 \quad \text{for} \quad i = 1, ..., n \quad (26)
\]

For these weights, investors are indifferent between joining the group and running a sole proprietorship. The factor \( e^{-z_0} (= J'(W_0)) \) implies that Equation (26) uniquely defines the weights in relative terms, not in absolute terms. For example, for \( n = 2 \), we get

\[
\frac{\lambda_2}{\lambda_1} = \left( \frac{W_{20}}{a_2} \right)^{\gamma_2} \left( \frac{W_{10}}{a_1} \right)^{\gamma_1}
\]

Investors’ utility weights are not proportional to the amounts they invest. Heterogeneity in risk preferences generates a skewed utility weighting (see Section 6 for a numerical example).

From Equation (26) it follows that the weighting normalizes the utility function by the investor’s shadow price of wealth so that each investor’s marginal utility of consumption for the rescaled utility equals 1. As such, our endogenous utility weights are similar to the Negishi (1960) welfare weights which have the unique feature of preserving the initial wealth distribution.\(^\text{11}\) By dividing each agent’s utility by their marginal utility of wealth at the optimum (a value that is smaller for wealthier agents than for otherwise identical poorer agents) the utility weights cause the marginal utility of consumption for relatively wealthy agents to be less at the maximum than for relatively poor agents. For the case

\(^{11}\)Of the social welfare functions consistent with Pareto optimality only weighted combinations of individual utilities with Negishi weights replicate market outcomes for given initial resource allocations. The Negishi (1960) weights are equal to the inverse of the marginal utility of wealth for each individual evaluated at the maximizing (equilibrium) resource allocation (see also Equation (37)).
of homogenous risk preferences, we obtain the following simpler rule, and the subsequent corollary:

\[ \frac{\lambda_i}{\lambda_1} = \left( \frac{W_{i0}}{W_{10}} \right)^\gamma \text{ if } \gamma_1 = \gamma_2 = \ldots = \gamma_n \equiv \gamma \]

**Corollary 5** Investors’ utility weights are proportional to the capital they invest if and only if all investors have logarithmic utility (i.e. \( \gamma = 1 \)).

The following proposition shows how risk aversion affects the concentration of the utility weights.

**Proposition 4** Suppose all investors have homogenous risk aversion and that the utility weights add up to 1. A higher (lower) level of risk aversion leads to more (less) concentrated utility weighting as measured by the Herfindahl index

\[ H = \sum_{j=1}^n \lambda_j^2 \]

That is, \( \frac{\partial H}{\partial \gamma} > 0 \) if \( \gamma_1 = \ldots = \gamma_n = \gamma \). Furthermore \( 0 < H < 1 \).

As all investors approach risk neutrality, the utility weights become uniform, i.e. \( \lim_{\gamma \to 0} H = 1/n \). On the other hand, as all investors become increasingly risk averse, small differences in capital contribution lead to large differences in utility weights. For extreme risk aversion (\( \gamma \to \infty \)), (almost) all utility weight is concentrated on the largest capital contributor, i.e. \( \lim_{\gamma \to \infty} H = 1 \).

The utility weights co-determine investors’ relative control on the investment decision. Investor \( i \)’s control weight \( \omega_{it} \) is increasing in her utility weight, i.e. \( \frac{\partial \omega_{it}}{\partial \lambda_i} > 0 \), holding all else equal (recall that the utility weights are themselves endogenous in our model).

The relation between the control weights and utility weights is time-varying and highly non-linear as can be inferred from the ratio of investors’ control weights:

\[ \frac{\omega_{jt}}{\omega_{it}} = \frac{a_j \lambda_j^{\frac{1}{\gamma_j}}}{a_i \lambda_i^{\frac{1}{\gamma_i}}} \cdot e^{z_t \left( \frac{1}{\gamma_j} - \frac{1}{\gamma_i} \right)} = \frac{a_j c_{jt}^*}{a_i c_{it}^*} \cdot \left( 1 - \tau_p \right) \left( \frac{1}{\gamma_i} - \frac{1}{\gamma_j} \right) \]

\[ (27) \]

\(^{12}\text{If investors have homogenous risk preferences and contribute the same amount of capital (i.e. } W_{10} = W_{20} = \ldots = W_{n0}, \text{ then the concentration index equals } H = 1/n \text{ for all values of } \gamma.\)
More (less) risk averse investors exert relatively more control in bad (good) times as 
\( z_t \to -\infty \) \((z_t \to +\infty)\). More control goes together with a relatively higher payout as 
reflected by the ratio \( c_{jt}/c_{it}^* \) in Equation (27). However, investors’ relative control rights 
\((\omega_{jt}/\omega_{it})\) are not equal to their relative cashflow rights \((c_{jt}^*/c_{it}^*)\), except if investors have 
the same risk preferences (i.e. \( \gamma_i = \gamma_j \), and therefore \( a_i = a_j \)). This illustrates the 
separation between cashflow rights and control rights. The following corollary results at once from Equation (27).

**Corollary 6** The balance of power within the group is constant over time if and only if 
all members have the same risk preferences. Control \((\omega_{it})\) is strictly proportional to the 
utility weights \((\lambda_i)\) if and only if all members have logarithmic utility (i.e. \( \gamma_i = 1 \) for 
i = 1, ..., n).

Investors who are relatively more risk averse exert very little influence on the firm’s investment and capital structure decision in good times. However, as the firm’s net worth declines, their control increases. This is similar to what happens in venture capital partnerships where preferred stockholders almost always have significant voting and control rights, especially when the firm is performing poorly.\(^{13}\)

Summarizing our results so far, we have shown that the utility weights are key drivers of the firm’s investment, financing and payout decisions, and of investors’ relative control on these decisions. As such the utility weights are important determinants of the firm’s internal governance structure. Although the utility weights are optimally fixed when the firm is set up, we will show below (see Section 4) that they remain optimal over time. As such there is no need to alter or adjust the firm’s governance structure as the business evolves and economic uncertainty unfolds.

\(^{13}\)Preferred stocks usually do not carry voting rights in public corporations. However, a frequent provision allows preferred shareholders to vote as a separate class whenever dividends have been in arrears for a specific period of time (normally four quarters) and to elect a specific number of directors.
3.6 Group risk aversion and financial policies

**Corollary 7** The group’s level of relative risk aversion (RRA) is endogenous and can be defined in terms of the value function as:

\[ RRA_G = -W_t \frac{J_{WW}}{J_W} = \frac{W(z_t)}{W'(z_t)} = \frac{\xi}{\sigma z_t^*} \]  

(28)

Using our comparative statics for \( \omega_t^* \) (see Section 3.1), it follows that:

\[ \frac{\partial RRA_G}{\partial W_t} \leq 0 \quad \text{and} \quad \lim_{z_t \to +\infty} RRA_G = \gamma_1 \quad \text{and} \quad \lim_{z_t \to -\infty} RRA_G = \gamma_n \]  

(29)

Hence, the group’s level of relative risk aversion decreases in its net worth, and converges to the level of the least (most) risk averse investor as net worth goes to infinity (zero). This explains the earlier described behavior of the firm’s investment and financing policy. More generally, it may also help explain why investors are very bullish during booms and extremely risk averse during recessions. Finally, \( RRA_G = \gamma \) if all investors have the same coefficient of risk aversion \( \gamma \).

3.7 Numerical examples and discussion

Figure 1 numerically evaluates the firm’s optimal payout and financing policies. The firm has 4 investors (or 4 classes of investors) with coefficients of relative risk aversion \( \gamma_1 = 0.5, \gamma_2 = 2, \gamma_3 = 3, \gamma_4 = 9 \), and with initial endowments \( W_{10} = W_{20} = W_{30} = W_{40} = 15 \)."
of risk aversion. In bad times, the more risk averse agents enjoy a higher payout level at the expense of less risk averse agents who take the hit. In good times, the least risk averse agent enjoys the upside, whereas all other agents’ payout is capped. The cap decreases with agents’ coefficient of risk aversion. This means that the payout function for each investor, except for the least risk averse one (represented by the dotted line), converges to a cap as wealth becomes large.\textsuperscript{15} From the figure, one can verify that the payout function for the least (most) risk averse agents is strictly convex (concave). For the other agents, the payout function is initially convex up to some inflection point, and concave thereafter.

Panel B plots the firm’s NDR as a function of net worth (thick solid line). Recall that a negative NDR ($D_t < 0$) corresponds to a firm with a net cash position. The NDR increases in the firm’s net worth when investors have heterogeneous risk preferences. Leverage is procyclical and converges towards 35.7\% as $W_t \rightarrow +\infty$. The firm delevers in economic downturns, reduces its holdings in risky assets and, in doing so, keeps debt risk free. The NDR approaches -0.91 as $W_t \rightarrow 0$. A negative NDR of -0.91 means that the firm’s net worth is invested for 91\% in cash and the remainder in risky assets. In line with Proposition 2 the firm’s NDR is bounded below and above by the optimal NDR for the most and least risk averse investor, respectively. Therefore, our model generates a large amount of intertemporal variation in leverage if investors’ risk preferences are widely dispersed.

The NDR is constant over time for a sole proprietorship (or when all investors have the same coefficient of risk aversion). For a single investor with coefficient of risk aversion equal to 0.5, 2, 3 or 9, the sole proprietorship adopts a constant net debt ratio of, respectively, 0.36, -0.61, -0.74 and -0.91.

Panel C plots the payout yields as a function of net worth. The optimal payout yield is constant when investors operate on their own (or all have the same coefficient of risk

\textsuperscript{15}In the Internet Appendix D, we plot investors’ payout, $c^*_{it}(W_t)$, and certainty equivalent claim values, $W^*(W_t)$, for a larger range of total net worth $W_t$ to illustrate the properties (such as the caps) of payout and claim value functions when $W_t$ is relatively high.
aversion). The four constant, horizontal lines represent the optimal payout yield for each sole proprietorship, which can either be calculated using the Merton (1969) model, or be obtained by calculating the ratio of the investor’s payout $c^*_i$ in the group and her certainty equivalent claim $W^*_i$ in the firm ($c^*_i/W^*_i$). Both calculations yield the same result. Note that the payout yield for sole proprietorships is not monotonic in the coefficient of risk aversion. In this particular example, the optimal payout yield is highest for investor 1 ($\gamma_1 = 0.5$). The payout yield of the group $c_t/W_t$ (given by the thick solid line) increases in net worth because investors are sufficiently risk averse. As $W_t$ goes to infinity the payout yield converges to the optimal yield of investor 1 ($\gamma_1 = 0.5$).

Panel D plots investors’ certainty equivalent claim value as a function of total net worth. When the firm was set up ($t = 0$), all four investors contributed 15 in net worth (i.e. $W_{i0} = 15$). The claim value for the most risk averse agent (solid line; $\gamma_4 = 9$) is strictly concave, the value of which slowly converges to 27.8 as total net worth becomes very large. The claim value of the least risk averse agent (dotted line; $\gamma_1 = 0.5$) is strictly convex. The claim values for investors with intermediate risk aversion (long dashed line for $\gamma_3 = 3$; short dashed lined for $\gamma_2 = 2$) are first convex and then concave. Except for the claim held by the least risk averse investor, all other claims converge to a bounded value as $W_t \to +\infty$. Even for investor 2, who is relatively less risk averse, the claim converges to a finite level. The bounded claims resemble preferred stock claims. In our model, preferred stocks can have different payout rate caps stacked in an echelon style fashion: investors with a higher level of risk aversion have a lower payout rate cap that is reached for lower levels of net worth.

Panel E plots investors’ control weights $\omega_{it}$. As net worth increases, less risk averse

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16 As $\gamma_i$ increases, the payout yield $(c^*_i/W^*_i = 1/\hat{a}_i)$ first increases and then decreases. For example, for another group of investors with $\gamma_1 = 0.5, \gamma_2 = 0.6, \gamma_3 = 3$, and $\gamma_4 = 9$, the optimal payout yield is highest for investor 2 (i.e. $\gamma_2 = 0.6$), not investor 1 ($\gamma_1 = 0.5$). As the total net worth goes to infinity, the payout yield still converges to the investor 1’s optimal yield, which is below the optimal yield for investor 2.

17 For example, the preferred stock valuation model by Emanuel (1983) produces claim values for cumulative preferred stock and non-cumulative preferred stock that are graphically very similar to ours (see Figures 3 and 4 on pages 1148 - 1149 in Emanuel (1983)), even though his valuation formulas are very different.
investors tend to exert more influence over the firm’s investment and financing policy. This relation can, however, be non-monotonic as illustrated by investor 2’s control weight which first increases and then decreases in the firm’s net worth. This highlights the complicated and non-trivial interactions in group decision making. For intermediate net worth levels (“normal” times) the members around the median level of risk aversion (investors 2 and 3) exert most influence (consistent with the findings of the experimental study by Ambrus, Greiner and Pathak (2015)).

Panel F plots the group’s level of relative risk aversion, \( RRA_G \). Group risk aversion is fairly low (around 0.51) for most levels of net worth, but spikes up dramatically once net worth drops below 100, converging to 9 as \( W_t \) goes to zero. During busts the group’s risk aversion and financial policies are determined (almost) entirely by the most risk averse investor.

4 Investors’ Life-time Utility and Group Irrelevance

In this section we compare a sole proprietor with an investor in a private firm in terms of their life-time utility and consumption flow. We first explore whether an individual investor can improve her relative utility weight by joining the firm at a later stage. Consider the scenario where investors do not join the firm at time 0, but at some future time \( t \). Assume that during the interval \([0, t]\) each single investor operates a sole proprietorship. We know from Merton (1969) that the net worth of each single individual investor, denoted by \( W_{it}^s \), evolves as:

\[
\frac{dW_{it}^s}{W_{it}^s} = \left[ \frac{\xi^2}{\gamma_i} + r - \frac{1}{a_i} \right] dt + \frac{\xi}{\gamma_i} dB_t \equiv \alpha_i dt + \sigma_i dB_t \tag{30}
\]

\[
W_{it}^s = W_{i0}^s e^{\left( (\alpha_i - \frac{\sigma_i^2}{2})t + \sigma_i B_t \right)} \tag{31}
\]

We know (see Equation (15)) that investor \( i \)’s utility weight when joining the partnership at time \( t \) is proportional to the inverse of her marginal utility of wealth, \((u'(W_{it}^s))^{-1} = \)
Note that the process (32) does not depend on the investor’s coefficient of risk aversion. One can then prove the following proposition:

**Proposition 5** Assume each investor can operate a sole proprietorship during the interval $[0, t]$. Whether investors join the group at time zero, or at some later time $t$, they join under the same relative utility weights. In particular:

$$
\frac{\lambda_i}{\lambda_j} = \left( \frac{W^*_{s,i}}{a_i} \right)^{\gamma_i} = \left( \frac{W^*_{s,j}}{a_j} \right)^{\gamma_j}
$$

(33)

This result might come as a surprise. How come the relative utility weights remain the same even though the net worth of less risk averse sole proprietors grows on average at a faster rate than the net worth of their more risk averse counterparts? The reason is that an investor’s utility weight depends on the inverse of her marginal utility of wealth. From Equation (32), we know that this quantity evolves in the same way for all sole proprietorships, irrespective of the investor’s coefficient of risk aversion. As a result, less risk averse investors cannot improve their utility weight by joining the proprietorship at a later point.

The question remains whether any investor enjoys a different payout flow as a sole proprietor compared with what she gets from a group-owned private firm.

**Proposition 6** If the group’s utility weights and the financial policies are optimally set as in Proposition 1, then the payout flow to each investor is the same whether she operates as a sole proprietor, or whether she is part of a heterogeneous group of investors in a private firm, i.e. $c^*_it = c^*_it$ for all $t$. Each investor’s imputed net worth stake $W^*_it$ within the private firm equals at all times the wealth $W^*_it$ she accumulates as a sole proprietor, i.e. $W^*_it = W^*_it$ for all $t$. 26
We obtain a striking group irrelevance result: it does not matter whether investors operate on their own, or as part of a group of \( n \) investors. Since all investors are subject to the same source of uncertainty and the same investment opportunity set, there are no benefits from diversification. Neither are there (dis)economies of scale from running a larger firm. However, in order to achieve this irrelevance result, the private firm has to adopt optimal dynamic investment and payout policies that are much more intricate than the ones adopted by a single proprietorship.

Let us compare next private firms with sole proprietorships in terms of the life-time utility they provide to investors. The life-time utility to investor \( i \) from joining the group is \( J_i(z_t) \). The aggregate utility for the group, \( J_G \), is therefore

\[
J_G(W(z_t)) = \sum_{i=1}^{n} J_i(z_t)
\]

(34)

In contrast, the aggregate utility generated by \( n \) sole proprietorships is given by:

\[
I(W_{1t},...,W_{2t}) = \sum_{i=1}^{n} I_i(W_{it}) = \sum_{i=1}^{n} a_i^{\gamma_i} \frac{W_{1t}^{1-\gamma_i}}{1-\gamma_i} - \frac{1}{\rho} \frac{1}{1-\gamma_i}
\]

(35)

The following corollary compares the aggregate utility of the group \( J_G(W) \) with the aggregate utility \( I(W_{1},...,W_{n}) \) of the \( n \) sole proprietorships.

**Corollary 8** Under the optimal policies, the aggregate life-time utility of \( n \) sole proprietorships with given endowments \( W_{i0} \) \((i = 1..n)\) equals the aggregate utility of the corresponding group of \( n \) investors in the private firm:

\[
J_G(W(z_0)) = \sum_{i=1}^{n} J_i(z_0) = \sum_{i=1}^{n} a_i^{\gamma_i} \frac{W_{10}^{1-\gamma_i}}{1-\gamma_i} e^{\frac{z_0(1-\gamma_i)}{\gamma_i}} - \frac{1}{\rho} \frac{1}{1-\gamma_i}
\]

(36)

\[
= \sum_{i=1}^{n} \frac{a_i^{\gamma_i}}{1-\gamma_i} W_{10}^{1-\gamma_i} - \frac{1}{\rho} \frac{1}{1-\gamma_i} = \sum_{i=1}^{n} I_i(W_{i0}) = I(W_{10},...,W_{n0})
\]

The marginal utility (shadow price) to the group of an additional dollar contributed by
member \( i \) is given by:

\[
\frac{\partial J_G}{\partial W_{i0}} = \left( \frac{a_i}{W_{i0}} \right)^{\gamma_i} = \frac{dJ_i(W(z_0))}{dW_{i0}} = \frac{dI_i(W(z_0))}{dW_{i0}} = u'_i((1 - \tau_p)c_{i0}^*) = \frac{dI_i(W_{i0})}{dW_{i0}}
\] (37)

The corollary states that the aggregate utility achieved by a group with \( n \) voluntary investors is the same as the aggregate utility by \( n \) sole proprietorships. This is a logical consequence of the group irrelevance result stated in Proposition 6.

As to be expected, the shadow price of an extra dollar of net worth, declines with the group’s total net worth \( W(z_t) \) already in place. Less standard is the result that the shadow price depends on which investor contributes the extra dollar. In particular a dollar contributed by the investor with the smallest (largest) weight adds the most (least) utility. The reason for this is that the utility weights \( \lambda_i \) voluntarily agreed by the investors do not maximize the group’s aggregate utility \( J_G \). Aggregate or “social” optimization requires that the marginal utilities of consumption (and therefore the shadow prices of wealth) are equalized across investors. From Equations (15) and (37) we know this is not the case. The social optimum requires that each investor gets the same utility weight, which is something not all investors would voluntarily agree to as it involves wealth transfers among investors that violate the participation constraints.

5 Personal and Corporate Taxes

This section considers the effect of personal and corporate taxes on investors’ payouts, claim values and utility weights. Consider first the effect of the personal tax rate \( \tau_p \). One can easily show that personal taxes do not affect the investment and payout policy of a sole proprietorship (i.e. \( \frac{\partial c^S_i}{\partial \tau_p} = \frac{\partial \omega^S_i}{\partial \tau_p} = 0 \)). However, personal taxes reduce after-tax consumption, and therefore the sole proprietor’s life-time utility (i.e. \( \frac{\partial I_i}{\partial \tau_p} \)).

From the group irrelevance result, we know that investors achieve the same payouts
from the private firm as from the sole proprietorship, provided that the utility weights incorporate investors’ participation constraint. Therefore, personal taxes should not affect investors’ payouts at startup because the utility weights are endogenous and internalize the prevailing tax rate. In what follows, we show that matters become more complicated if investors face an unexpected tax change after the firm and its governance structure (i.e. utility weights) are set up.

The first-order conditions for investors’ payouts are given by:

\[ u_i'((1 - \tau_p)c_i^*) = \frac{J'(W_0)}{\lambda_i} = \frac{e^{-z_0}}{\lambda_i} = \frac{dI_i(W_{i0})}{dW_{i0}} = \frac{dJ_i(W_0)}{dW_0} \quad \text{for } i = 1, \ldots, n \]  

(38)

At startup the utility weights are set such that each investor’s marginal utility of consumption equals her shadow price of capital. The latter is determined by the marginal indirect utility that the investor derives from investing in a sole proprietorship (i.e. her outside option). With heterogenous preferences and heterogenous capital endowments, the shadow price of capital varies across investors. Consequently also the utility weights vary across investors since

\[ \frac{\lambda_i}{\lambda_j} = \frac{\frac{dI_i(W_{i0})}{dW_{i0}}}{\frac{dI_j(W_{j0})}{dW_{j0}}} = \frac{a_j^{\gamma_j} W_{j0}^{-\gamma_j}}{a_i^{\gamma_i} W_{i0}^{-\gamma_i}} = \frac{(1 - \tau_p)^{1-\gamma_j} \hat{a}_j^{\gamma_j} W_{j0}^{-\gamma_j}}{(1 - \tau_p)^{1-\gamma_i} \hat{a}_i^{\gamma_i} W_{i0}^{-\gamma_i}} = \frac{(1 - \tau_p)^{\gamma_i - \gamma_j} \hat{a}_j^{\gamma_j} W_{j0}^{-\gamma_j}}{\hat{a}_i^{\gamma_i} W_{i0}^{-\gamma_i}} \]  

(39)

Equation (39) shows that the relative utility weights are inversely related to the ratio of investors’ shadow prices of capital. All else equal, an investor with a lower shadow price of investing in the sole proprietorship gets a higher utility weight to ensure that the weighted shadow prices \( \lambda_i \frac{dI_i(W_{i0})}{dW_{i0}} \) are equalized to \( J'(W_0) \) across investors. This implies (see Eq. (39)) that higher personal taxes shift utility weight from the more risk averse investor (e.g. the venture capitalist) to the less risk averse investor (e.g. the entrepreneur). These effects are entirely driven by investors’ heterogeneous preferences:

**Corollary 9** **If investors have homogenous risk preferences then the relative utility weights only depend on the investors’ initial capital contribution and the common coefficient of**
RRA. The tax rates (and any other model parameter for that matter) do not influence the relative utility weights.

Taxation rates play an important role for the utility weights and the firm’s governance if investors have heterogeneous preferences.

**Proposition 7** Assume that the utility weights are normalized to add up to 1. There exists critical risk aversion thresholds \( \hat{\gamma}, \bar{\gamma} \in (\gamma_1, \gamma_n) \) such that an investor’s utility weight decreases in the personal (corporate) tax rate if and only if her coefficient of risk aversion exceeds \( \hat{\gamma} (\bar{\gamma}) \), i.e.:

\[
\exists \hat{\gamma}, \bar{\gamma} \in (\gamma_1, \gamma_n) : \frac{\partial \lambda_i}{\partial \tau_p} \geq (\leq) 0 \iff \gamma_i \leq (\geq) \hat{\gamma} \quad \text{and} \quad \frac{\partial \lambda_i}{\partial \tau_c} \geq (\leq) 0 \iff \gamma_i \leq (\geq) \bar{\gamma}
\]

\( \hat{\gamma} \) and \( \bar{\gamma} \) are defined in the Appendix by Equations (71) and (73), respectively.

Let us examine next how the utility weights depend on the model parameters (such as taxation rates) when investors have heterogeneous preferences. Consider investors’ payout from the private firm \( c_{it}^* \):

\[
c_{it}^* = (1 - \tau_p)^{\frac{1}{\gamma_i} - 1} \left( J'(W_t) \right)^{-\frac{1}{\gamma_i}} \lambda_i^{\frac{1}{\gamma_i}} = (1 - \tau_p)^{\frac{1}{\gamma_i} - 1} e^{\frac{\gamma_i}{\gamma}} \lambda_i^{\frac{1}{\gamma}}
\]

\[
= (1 - \tau_p)^{\frac{1}{\gamma} - 1} \left[ \frac{W_t \lambda_i^{\frac{1}{\gamma}}}{(1 - \tau_p)^{\frac{1}{\gamma} - 1} \frac{1}{a} \sum_{i=1}^{n} \lambda_i^{\frac{1}{\gamma}}} \right] = \frac{W_t W_{it0}}{W_0} \quad \text{if} \quad \gamma_i = \gamma \quad \text{for all} \quad i = 1, ..., n \tag{41}
\]

Payout is determined by three factors. The first factor \( ((1 - \tau_p)^{\frac{1}{\gamma} - 1}) \) reflects the effect of \( \tau_p \) on the investor’s marginal utility of consumption. The second \( (J'(W)^{-\frac{1}{\gamma}}) \) and third factor \( (\lambda_i^{\frac{1}{\gamma}}) \) determine the investor’s shadow price of capital from investing in a sole proprietorship. Equation (41) shows that under homogenous preferences the proportional effect of personal taxes on the investor’s marginal utility is exactly offset by the same proportional effect of \( \tau_p \) on the group’s marginal utility \( J'(W_t) \), without the need for an adjustment in the investor’s utility weight \( \lambda_i \). When investors have homogenous risk preferences, each investor’s payout is a simple linear function of the firm’s total net worth
$W_t$ with investor $i$ getting a fraction $W_{i0}/W_0$ of total payout.

When investors have heterogeneous preferences, each investor’s marginal utility of consumption no longer moves in lockstep with other investors’ marginal utility. Consequently, investors’ utility weights act as balancing factors to ensure that each investor’s marginal utility of consumption equals her shadow price of capital. In particular, changes in the utility weights ensure that:

$$\frac{dc^*_i}{d\tau_p} \bigg|_{W_{i0}} = \frac{\partial c^*_i}{\partial \tau_p} + \frac{\partial c^*_i}{\partial z_0} d\tau_p + \frac{\partial c^*_i}{\partial \lambda_i} d\lambda_i + \frac{dc^*_i}{d\tau_p} = 0$$

The first term reflects how individual payout changes when $\tau_p$ changes, holding all else constant. The second term shows how $\tau_p$ indirectly affects payout through changes in the group’s shadow price of capital. The third term captures the sensitivity of the firm’s governance structure (i.e. the utility weights) to changes in $\tau_p$. The third term ($\frac{dc^*_i}{d\lambda_i} d\lambda_i$) is zero only when all investors have the same coefficient of RRA. Since personal taxes do not affect the payouts of investors’ outside option, and since investors’ participation constraint binds, $\tau_p$ does not affect investors’ payout from the private firm (i.e. $\frac{dc^*_i}{d\tau_p} \bigg|_{W_{i0}} = 0$).

Recall that the utility weights are determined and fixed at startup. Consider now the following thought experiment. Imagine that sometime after startup there is an unanticipated change in the personal tax rate $\tau_p$. What would be the effect on the firm’s payout policy and investors’ claim values? We know that $\tau_p$ does not affect the payout of a sole proprietorship. The same is, however, not true for the private firm if investors’ utility weights are fixed and do not adjust to the tax policy change.

**Proposition 8** An unexpected change in the personal tax rate after startup would lead to wealth transfers among investors if the utility weights are fixed. The effect of an
unanticipated change in $\tau_p$ on investors’ payout $c^*_t$ and net worth stake $W_{it}$ is given by:

$$\frac{\partial c^*_t}{\partial \tau_p}, \frac{\partial W_{it}}{\partial \tau_p} < 0 < \frac{\partial c^*_{nt}}{\partial \tau_p}, \frac{\partial W_{nt}}{\partial \tau_p}$$

and

$$\frac{\partial c^*_{it}}{\partial \tau_p}, \frac{\partial W_{it}}{\partial \tau_p} \leq (\geq) 0 \iff W_i \leq (\geq) W(\hat{z}_i) \text{ for } i = 2, \ldots, n - 1$$

where $\hat{z}_i$ is the solution to

$$\sum_{j=1}^{n} \left( \frac{\gamma_i}{\gamma_j} - 1 \right) a_j (\lambda_j e^{\hat{z}_i})^{\frac{1}{\gamma_j}} = 0 \quad (43)$$

with $W(\hat{z}_2) \geq W(\hat{z}_3) \geq \ldots W(\hat{z}_{n-2}) \geq W(\hat{z}_{n-1})$

The proposition shows that changes in the personal tax rate lead to a redistribution of investors’ payouts and net worth stakes if the firm’s internal governance structure (i.e. the utility weights) is fixed. The least (most) risk aversive investor always loses (gains) from an increase in the personal tax rate. For all other investors, there exists an investor-specific critical threshold $W(\hat{z}_i)$ such that an increase in $\tau_p$ leads to a decrease (increase) in investor $i$’s payout for total net worth levels below (above) $W(\hat{z}_i)$. Hence, personal tax hikes tend to reduce payout of small (i.e. low net worth) firms and increase payout of large firms. Moreover, the critical thresholds decline in investor’s risk aversion so that unexpected hikes in the personal tax rate are more likely to have an adverse (positive) effect on less (more) risk averse investors.

Consider next the effect of corporate taxes. One can show that for sole proprietorships corporate taxes increase investment in risky assets (i.e. $\frac{\partial \omega^S_i}{\partial \tau_c} > 0$), and increase (decrease) payouts of investors with a coefficient of RRA below (above) one (i.e. $\frac{\partial c^S_i}{\partial \tau_c} \geq (\leq) 0 \iff \gamma_i \leq (\geq) 1$). Corporate taxes reduce the sole proprietor’s life-time utility (i.e. $\frac{\partial I_i}{\partial \tau_c} < 0$). Our group irrelevance result implies that at startup (i.e. when the utility weights are fixed) the future payouts to investors in the private firm are the same as what they get from a sole proprietorship, irrespective of the corporate tax rate. The effect of corporate tax changes on investors’ payouts is more complicated after the utility weights are fixed. From Equation (40) it follows that $\tau_c$ affects $c^*_t$ only through $z_t$ once the utility weights are fixed. Furthermore, since $W_{it} = \hat{a}_i c^*_t$ and $\hat{a}_i$ depends on $\tau_c$, it follows that the
comparative statics results for $c^*_it$ do not necessarily carry over to $W_{it}$, as is shown in the below proposition.

**Proposition 9** For given utility weights $\lambda_i$ and net worth $W_t$, the effect of the corporate tax rate $\tau_c$ on investors’ net worth stake $W_{it}$ and payout $c^*_it$ is as follows:

\[
\frac{\partial W_{it}}{\partial \tau_c} < 0 < \frac{\partial W_{nt}}{\partial \tau_c}
\]

\[
\frac{\partial W_{it}}{\partial \tau_c} \leq (\geq) 0 \iff W_t \leq (\geq) W(z_{iW}) \text{ for } i = 2, \ldots, n - 1
\]

If $\gamma_i \geq (\leq) 1$ for all $i = 1, \ldots, n$ then $\frac{\partial c^*_it}{\partial \tau_c} \leq (\geq) 0$

If $\gamma_1 \leq \ldots \leq \gamma_k \leq 1 \leq \gamma_{k+1} \leq \ldots \leq \gamma_n$, then $\frac{\partial c^*_it}{\partial \tau_c} \leq (\geq) 0 \iff W_t \leq (\geq) W(z_c)$

The thresholds $z_{iW}$ and $z_c$ are defined by Equations (81) and (85) in the Appendix.

Analogous to the sole proprietorship case, higher corporate taxes increase (reduce) investors’ payouts if the coefficient of RRA is below (above) one for all investors. If the coefficients of RRA are spread above as well as below 1 then higher corporate taxes reduce (increase) payouts of all investors if the firm’s net worth is below (above) a common threshold $W(z_c)$.

The results for the net worth stakes $W_{it}$ mirror the ones we obtained for personal taxation: higher corporate taxes shift net worth from the least risk averse to the most risk averse investor. The net worth of the other investors ($i = 2, \ldots, n - 1$) decrease (increase) in the corporate tax rate if total net worth is below (above) some investor-specific threshold $W(z_{iW})$.

In summary, if the utility weights are not yet fixed (i.e. at startup when investors join the firm), then personal taxes do not affect investors’ payouts and net worth claims, whereas corporate taxes increase (decrease) payouts of investors with a coefficient of RRA below (above) one. Once the utility weights are fixed (i.e. after startup) then an unexpected change in the personal or corporate tax rate has a redistributing effect on
investors’ payout and net worth stake. Tax rises increase (decrease) the net worth of the most (least) risk averse investor. For investors with intermediate levels of risk aversion, tax rises reduce (increase) their net worth if the firm’s total net worth is below (above) some threshold.

6 Utility Weights: A Numerical Example

We conclude this section by providing a numerical example that calculates investors’ utility weights and life-time utility. Table 1 considers 10 investors with different coefficients of risk aversion ranging from $\gamma_1 = 0.5$ to $\gamma_{10} = 9$. All investors have the same initial endowment. Panel A considers poorly endowed investors who each contribute 5 (i.e. $W_{i0} = 5$ for $i = 1, ..., n$). Panels B and C consider the cases of intermediate ($W_{i0} = 10$) and high ($W_{i0} = 50$) initial capital contributions, respectively.\(^{18}\)

Low initial capital contributions (Panel A) confer an advantage to investors who are less risk averse as the two least risk averse investors capture together 72% of the utility weight in the benchmark case ($\lambda_1 = 0.365$ and $\lambda_2 = 0.355$). The utility weights and life-time utility both decline in investors’ coefficient of risk aversion. The utility weights of the three most risk averse investors are almost zero.

The utility weights are an inverted U-shaped function of risk aversion for medium levels of capital contribution. They are still highly skewed with the four least risk averse investors taking more than 80% of the total weight. Life-time utility is decreasing in risk aversion. The disparity in life-time utility is as significant, with investor 10 and investor 1 achieving the lowest (-160.80) and highest (0.46) life-time utility, respectively.\(^{19}\)

The utility weights are increasing in the coefficient of risk aversion if investors’ initial

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\(^{18}\)The benchmark parameter values are: $\rho = 0.1, \mu = 0.12, \sigma = 0.3, \tau_p = \tau_c = 0$ and $r = 0.05$.

\(^{19}\)Since $U_i(c_i) = (c_i^{1-\gamma_i} - 1)/(1 - \gamma_i)$, utility levels can be negative. However, the utility function is well behaved in that it is monotonic, continuous and differentiable in $\gamma_i$ for all $\gamma_i \geq 0$. Life-time utilities for investors with $\gamma_i < 1$ can be negative due to the constant term $-1/(\rho(1 - \gamma_i))$ in the life-time utility function.
capital contribution is high ($W_{i0} = 50$), with the most risk averse agent taking 80% of the utility weight ($\lambda_{10} = 0.8$) and the least risk averse investor having a weight close to zero. Although life-time utility is still declining in risk aversion, the degree of inequality is significantly smaller than in the case where investors are poorly endowed.

More generally, one can prove (see Internet Appendix B) that if $W_{10} = \ldots = W_{n0} \equiv w_0$, there exist initial individual wealth thresholds $\underline{w}_0$ and $\overline{w}_0$ (with $\underline{w}_0 < \overline{w}_0$) for which the utility weight, $\lambda_i$, is a monotonically decreasing (increasing) function of the coefficient of risk aversion, $\gamma_i$, if each investor’s initial wealth contribution $w_0$ is below (above) $\underline{w}_0$ ($\overline{w}_0$).\footnote{We normalize the utility weights so that they add up to 1.} The utility weight is an inverted U-shaped function of the coefficient of risk aversion, $\gamma_i$, if each investor’s initial wealth contribution falls within the interval $(\underline{w}_0, \overline{w}_0)$.

The table also provides results for a different level of asset return volatility ($\sigma$), risk-free rate ($r$), personal tax rate ($\tau_p$) and corporate tax rate ($\tau_c$). The utility weight increases (decreases) with volatility for investors with low (high) risk aversion. For example, increasing volatility from 0.3 to 0.4 in Panel B (i.e. $W_{i0} = 10$) increases the utility weight of investor 1 ($\gamma_1 = 0.5$) from $\lambda_1 = 0.227$ to $\lambda_1 = 0.253$. Higher volatility unambiguously reduces the life-time utility of all investors.

The risk free rate of interest $r$ enters the formula for $\lambda_i$ in a more intricate fashion than volatility $\sigma$.$^{21}$ Increasing the risk free rate $r$ reduces (increases) the life-time utility of less (more) risk averse investors because they are net borrowers (savers). A higher $r$ raises (reduces) the utility weight of investors with high (low) risk aversion.

Finally, as shown in Proposition 7, increasing the personal or corporate tax rate, increases (decreases) the utility weights of investors with a low (high) level of risk aversion. For example, increasing $\tau_p$ from 0 to 0.15 when $W_{i0} = 10$ (panel B), increases the utility weight of the least risk averse investor from 0.227 to 0.265. Any increase in $\tau_p$ or $\tau_c$ unambiguously reduces the life-time utility of all investors.

$^{20}$Both the volatility ($\sigma$) and the drift ($\mu$) of the asset return affect the utility weights only through the Sharpe ratio $\xi \equiv (\mu - r)/\sigma$. Consequently the effect of $\mu$ on the $\lambda_i$ is analogous (but in the opposite direction) to the effect of $\sigma$. 
7 Empirical Implications and Conclusions

Our paper provides empirical implications and conclusions that are relevant for a variety of literatures in financial economics.

7.1 Risk aversion and group decision making

The literature offers two competing hypotheses for group decisions: the group shift hypothesis (e.g. Moscovici and Zavalloni (1969); Kerr (1992)) and the diversification of opinions hypothesis (e.g. Sah and Stiglitz (1986, 1988)). Although the latter hypothesis appears to enjoy strong empirical support (see Bär, Kempf and Ruenzi (2011)), it struggles to explain the puzzling observation that the average group is more (less) risk averse than the average individual in high (low) risk situations. Our model provides a possible explanation by showing that group decisions are not merely a weighted average of individuals’ preferred decisions but, crucially, the control weights vary over time. In our model, more weight shifts towards the least risk averse agent as the firm grows during booms, whereas in busts increased weight shifts towards the most risk averse agent. The group’s risk aversion is determined primarily by the most risk averse agent when net worth falls to critically low levels. During “normal” times the member with the median level of risk aversion exerts most influence (consistent with the findings of the experimental study by Ambrus, Greiner and Pathak (2015)). As such our model reconciles the diversification of opinions hypothesis with the group shift hypothesis.

7.2 Firm Leverage

Our model predicts that small startups are initially all-equity financed holding a net cash balance. A line of credit is gradually introduced to finance growth as retained earnings and net worth accumulate. To our knowledge, we are the first to show that heterogeneity in investors’ risk preferences can lead to procyclical leverage, and be a significant source
of intertemporal variation in a firm’s debt ratio.

Our model results in an optimal capital structure that includes safe (or no) debt, preferred stock and equity. The model generates a wide range of time-varying net debt ratios between -1 and +1 without relying on taxes, bankruptcy costs, asymmetric information or other frictions traditionally invoked by standard capital structure theories. Our findings may apply not only to private firms but also to public corporations that prohibit managers from trading the stock they own in the firm. Chava and Purnanandam (2010) show that CEOs’ risk preferences affect leverage and cash-holding policies, whereas CFOs’ risk preferences are relatively more important in explaining debt maturity structure and accrual decisions. They conclude that “closer attention should be paid to the risk preferences and attitudes of managers to better understand the corporate financial decision making”.

7.3 Capital structure and preferred stock

Our model predicts that equity claims are tailored to investors’ risk preferences, with less (more) risk averse investors having claims that tend to be relatively more convex (concave) in the firm’s net worth. Importantly, an investor’s optimal contract not only depends on her own level of risk aversion, but also on the risk preferences of the other investors. An investor may opt for a contract that is relatively performance sensitive when matched with more risk averse co-investors, and choose a contract that is relatively performance insensitive when matched with less risk averse investors. These findings generate new empirical hypotheses regarding the capital structure of private firms.

Our model generates an optimal capital structure that contains a variety of claims resembling preferred stock. We have shown that these contracts allow efficient risk sharing among a group of investors with heterogeneous risk preferences who cannot trade their claims. The contracts we derive cap the payout of more risk averse investors at lower levels in good times, but give the more risk averse investors priority in terms of payout in
bad times. This novel rationale for using preferred stock may help explain the common occurrence and huge variety of preferred stock contracts that we observe in the venture capital sector (see Kaplan and Strömberg (2003)). Another distinguishing feature of venture capital financings is that they allow VCs separately to allocate cash flow rights and control rights. Our model separates investors’ cashflow rights from their control rights, and shows that both can be time-varying.

7.4 Governance and control

We show that the governance structure implemented at startup is a function of the prevailing model parameters, such as the personal and corporate taxation rates. The private firm’s governance structure at startup ensures that no single investor is put at a disadvantage compared to what she could achieve in a sole proprietorship. Given that investors cannot withdraw their capital from the private firm, unanticipated tax changes after startup can lead to wealth transfers among investors if the governance structure is not recalibrated to the new tax environment. This suggests that taxation may play an important role in the choice of legal and governance structures that are adopted for private firms.

Our model shows that increasing heterogeneity in investors’ risk preferences leads, all else equal, to higher utility weight concentration. As a result, investors who start ex ante with equal wealth endowments may ex post end up with highly unequal say or influence on the firm’s decision making. We show that an investor’s relative influence (as reflected by her control weight) over the investment decision is usually a non-linear function of her utility weight. Utility weights, control rights and cash flow rights are proportional to the amount of capital invested if and only if all investors have logarithmic utility ($\gamma = 1$). With homogenous risk preferences, the Herfindahl index for the utility weights of $n$ investors converges to $1/n$ or 1, if all investors become risk neutral or infinitely risk averse, respectively. In the latter case (almost) all utility weight is concentrated in
the hands of the firm’s largest capital contributor. Our model shows that an individual investor’s control over the group’s financial policies is time-varying with more (less) risk averse investors assuming more control in bad (good) times. This mirrors what happens in the venture capital industry where VCs (entrepreneurs) assume increasing control when performance is bad (good).

7.5 Dynamics of group decisions and future research

Our paper highlights the dynamics of optimal group policies and generates a rich diversity of corporate liabilities in the firm’s optimal capital structure. These features resolve the differences in preferences of investors in small private firms who cannot trade their claims. Although it is optimal for a sole proprietor with CRRA to adhere to a constant net debt ratio (NDR) and a constant payout ratio, a group of investors that jointly own a private firm adopt an optimal NDR and payout yield that are time-varying (procyclical). Our model also shows that control within the group varies over time in response to shocks. Although each investor has a constant coefficient of RRA, the group’s level of RRA is a time-varying weighted average of investors’ coefficient of risk aversion. This causes the group’s risk aversion to be decreasing in the firm’s net worth.

For the group to replicate the aggregate welfare of the sole proprietorships and for debt to remain risk-free, the optimal financial policies require continuous rebalancing. A static policy or discrete-time rebalancing reduce aggregate welfare, and the firm’s debt may no longer be risk-free if net worth drops too much before rebalancing takes place. Despite these complexities, we managed to obtain tractable, closed-form solutions for the dynamic model. Future research might alter or relax some of our model assumptions. For example, one might endow investors with different investment opportunity sets or with individual idiosyncratic shocks, allowing groups to generate synergies or diversification benefits. This will destroy the group irrelevance results and raise the question as to how these synergies or benefits are shared among investors over time.
Appendix

Proof of Proposition 1

We solve the constrained optimization problem in two steps. First, we derive the optimal investment and payout policies by taking the weights \( \{(\lambda_1, ..., \lambda_n) | \lambda_i > 0, i = 1, ..., n\} \) as exogenously given. Second, we endogenize the utility weights by imposing the participation constraints (7). For any positive weights, the group’s indirect value function \( J(W_t) \) satisfies the following Hamilton–Jacobi–Bellman (HJB) equation (see the dynamic programming method described in Dixit and Pindyck (1994))

\[
\rho J(W_t) = \max_{\{c_{it}, \omega_t\}} \sum_{i=1}^{n} \lambda_i u_i((1 - \tau_p)c_{it}) + \left[(\omega_t(\mu - r) + r) W_t - \sum_{i=1}^{n} c_{it}\right] J'(W_t) + \frac{1}{2} \sigma^2 \omega_t^2 \sum_{i=1}^{n} J''(W_t)
\]

(44)

The first-order conditions are

\[
\omega_t^* = -\frac{\xi}{\sigma} \frac{J'(W_t)}{W_t J''(W_t)} \quad \text{and} \quad c_{it}^* = \left(\frac{J(W_t)}{\lambda_i}\right)^{-\frac{1}{\gamma_i}} (1 - \tau_p)^{-\frac{1}{\gamma_i} - 1}
\]

(45)

Substituting the first-order conditions into the HJB yields

\[
\sum_{i=1}^{n} \left[ \frac{\gamma_i}{1 - \gamma_i} \frac{\lambda_i^{\frac{1}{\gamma_i}}}{\left(\frac{J'(W_t)}{1 - \tau_p}\right)^{1 - \frac{1}{\gamma_i}}} - \frac{\lambda_i}{1 - \gamma_i} \right] + r W_t J'(W_t) - \frac{1}{2} \sigma^2 \frac{J'(W_t)^2}{J''(W_t)} - \rho J(W_t) = 0
\]

(46)

The non-linear ordinary differential equation cannot be solved analytically in terms of \( W_t \). However, tractable results are made possible by introducing an auxiliary state variable \( z_t \):

\[
z_t \equiv -\ln(J'(W_t)) \Rightarrow J'(W_t) = e^{-z_t} \quad \text{and} \quad J''(W_t) = -\frac{e^{-z_t}}{W'(z_t)}
\]

(47)

Substituting \( J'(W_t) \) and \( J''(W_t) \) into Equation (46) leads to

\[
\sum_{i=1}^{n} \left[ \frac{\gamma_i}{1 - \gamma_i} \frac{\lambda_i^{\frac{1}{\gamma_i}}}{\left(\frac{e^{-z_t}}{1 - \tau_p}\right)^{1 - \frac{1}{\gamma_i}}} - \frac{\lambda_i}{1 - \gamma_i} \right] + r W(z_t) e^{-z_t} + \frac{1}{2} \sigma^2 W'(z_t) e^{-z_t} - \rho J(W(z_t)) = 0
\]

40
Differentiating the transformed ODE with respect to \( z_t \), and then dividing both sides by \( e^{-z_t} \) gives the following second-order linear ODE in \( z_t \):

\[
\sum_{i=1}^{n} \lambda_i^\frac{1}{n} (1 - \tau_p)^{\frac{1}{n}} - 1 e^{\frac{z_t}{n}} + \left( r - \frac{1}{2} \xi^2 - \rho \right) W''(z_t) + \frac{1}{2} \xi^2 W'''(z_t) - r W(z_t) = 0 \tag{48}
\]

The solution to the above ODE has two components: a particular solution \( W_p \) and a complementary solution \( W_c \), i.e. \( W = W_p + W_c \). One can conjecture and verify that the particular solution is given by

\[
W_p(z_t) = \sum_{i=1}^{n} G_i e^{\frac{z_t}{n}} \quad \text{with} \quad G_i = \frac{\lambda_i^{\frac{1}{n}} \gamma_i^2 (1 - \tau_p)^{\frac{1}{n}} - 1}{r \gamma_i^2 + (\rho + \frac{\xi^2}{2} - r) \gamma_i - \frac{\xi^2}{2}} \equiv a_i \lambda_i^{\frac{1}{n}} \tag{49}
\]

The complementary solution is of the form \( W_c(z_t) = H_1 e^{b_+ z_t} + H_2 e^{b_- z_t} \), where \( H_1 \) and \( H_2 \) are constants to be determined and \( b_\pm = (\rho + \frac{\xi^2}{2} - r) \pm \sqrt{(\rho + \frac{\xi^2}{2} - r)^2 + 2 \xi^2 r} \). It can be shown that \( b_+ > 0 \) and \( b_- < -1 \). The particular solution captures the expected lifetime utility under a particular payout and investment policy. The complementary solution captures the value from growth options and abandonment options, which investors do not have. Therefore, the complementary part must be zero, i.e. \( H_1 = H_2 = 0 \), as otherwise a bubble component would be added that explodes for the negative root when \( z_t \to -\infty \) (\( W_t \to 0 \)), and for the positive root when \( z_t \to +\infty \) (\( W_t \to +\infty \)). Thus,

\[
W(z_t) = \sum_{i=1}^{n} G_i e^{\frac{z_t}{n}} = \sum_{i=1}^{n} a_i (\lambda_i e^{z_i})^{\frac{1}{n}} \tag{50}
\]

A solution to the optimization problem also needs to satisfy the transversality condition and the feasibility condition, which are given by

\[
D_i \equiv r \gamma_i^2 + (\rho + \frac{\xi^2}{2} - r) \gamma_i - \frac{\xi^2}{2} > 0 \quad \text{and} \quad \rho > (1 - \gamma_i) \left( \frac{\xi^2(1 + \gamma_i)}{2 \gamma_i} + 2 \right) \tag{51}
\]
Examine the two conditions together suggests that for $i = 1, ..., n$,

$$
\gamma_i > \gamma^* \equiv - (\rho - r) + \sqrt{(\rho - r)^2 + 2\xi^2 (r + \frac{\xi^2}{2})} \over 2(r + \frac{\xi^2}{2})
$$

(52)

In other words, for the least risk averse investor $\gamma_1 > \gamma^* \in (0, 1)$. Substituting $W(z_t)$ into Equation (48) gives the group’s indirect utility function (14), while substituting into the first-order conditions (45) gives (9) and (10).

We further denote the right-hand side of (44) as $\phi(W_t)$. One could then verify the second-order conditions by showing that

$$
\left. \frac{\partial^2 \phi}{\partial \omega^2} \right|_{c_i = c_i^*, \omega = \omega^*} = - \frac{\sigma^2 W(z_t)^2 e^{-z_t}}{W'(z_t)} < 0 \quad \left. \frac{\partial^2 \phi}{\partial c_i^2} \right|_{c_i = c_i^*, \omega = \omega^*} = - \lambda_i \gamma_i(c_i^*)^{-1 - \gamma_i} < 0
$$

As $\frac{\partial^2 \phi}{\partial \omega \partial c_i} = 0$ and $\frac{\partial^2 \phi}{\partial c_i \partial c_j} = 0$ for $i \neq j$, the determinants of the leading principle minors of the Hessian matrix are: $\det(H_1) = \frac{\partial^2 \phi}{\partial \omega^2}$ and $\det(H_k) = \frac{\partial^2 \phi}{\partial c_i^2} \prod_{i=1}^{k-1} \frac{\partial^2 \phi}{\partial c_i^2}$ for $k > 1$. The sequence alternates in sign which completes the verification.

The results above hold for any non-negative weights $\{(\lambda_1, ..., \lambda_n)|\lambda_i > 0, i = 1, ..., n\}$. In the second step, we introduce the participation constraints to endogenize the weights to guarantee all investors’ participation. Investor $i$’s participation constraint (Eq. (7)) is satisfied if $a_i \lambda_i \gamma_i e^{z_0} \geq W_i$. Combining all investors’ participation constraints yields:

$$
\sum_{i=1}^{n} a_i \lambda_i \gamma_i e^{z_0} \geq \sum_{i=1}^{n} W_i = W_0
$$

(53)

We know from Equation (50) that the above constraint is always satisfied as an equality. Therefore, under the group’s optimal policies $(c_1^*, ..., c_n^*, \omega^*)$, each investor’s individual participation constraint must be binding. In other words, each investor is indifferent between joining the group or going it alone. As a result:

$$
a_i (\lambda_i e^{z_0})^{1 \over \gamma_i} = W_i \quad \text{for} \quad i = 1, ..., n
$$

(54)
Rearranging the above equality gives investor $i$'s voluntary participation weight: $\lambda_i = \left( \frac{W_{i0}}{a_i} \right)^{\gamma_i} e^{-\tau_0}$. Moreover, investor $i$'s marginal utility from payout $c_i$ is given by

$$u'_i((1 - \tau_p)c_i) = \left[ (\lambda_i e^{z_0})^{\frac{1}{\gamma_i}} \right]^{-\gamma_i} = (\lambda_i e^{z_0})^{-1} = \left( \frac{a_i}{W_{i0}} \right)^{\gamma_i} = \frac{dI_i(W_{i0})}{dW_{i0}}$$

which shows Equation (15) and completes the second step of the proof. □

Proof of Proposition 2

As shown in Equation (9) in Proposition 1,

$$\omega^*_t = \frac{\xi}{\sigma} \sum_{i=1}^{n} \frac{1}{\gamma_i} \omega_i(z_t) = \frac{\xi}{\sigma} \left[ \frac{1}{\gamma_1} \omega_1(z_t) + \sum_{i=2}^{n-1} \frac{1}{\gamma_i} \omega_i(z_t) + \frac{1}{\gamma_n} \omega_n(z_t) \right]$$

Rewrite $\omega_i(z_t)$ as

$$\omega_i(z_t) = \frac{G_i e^{z_t}}{\sum_{j=1}^{n} G_j e^{z_j}} = \left( 1 + \sum_{j \neq i} \frac{a_j \lambda_j}{a_i \lambda_i} e^{z_j} \right)^{-1} \equiv \frac{1}{1 + \sum_{j \neq i} k_j(z_t)}$$

If $\gamma_i > (\gamma_j$, lim$_{z_t \to +\infty} k_j(z_t) = +\infty (0)$ and lim$_{z_t \to -\infty} k_j(z_t) = 0 (+\infty)$, which implies

$$\lim_{z_t \to +\infty} \omega_1(z_t) = 1; \quad \lim_{z_t \to -\infty} \omega_1(z_t) = 0; \quad \lim_{z_t \to +\infty} \omega_n(z_t) = 0; \quad \lim_{z_t \to -\infty} \omega_n(z_t) = 1$$

$$\lim_{z_t \to +\infty} \omega_i(z_t) = 0; \quad \lim_{z_t \to -\infty} \omega_i(z_t) = 0 \quad \text{for } i = 2, \ldots, n - 1$$

Aggregating according to Equation (56) shows

$$\lim_{z_t \to +\infty} \omega^*(z_t) = \frac{\xi}{\sigma} \frac{1}{\gamma_1} \quad \text{and} \quad \lim_{z_t \to -\infty} \omega^*(z_t) = \frac{\xi}{\sigma} \frac{1}{\gamma_n}$$

The inequality (16) then follows directly from Equation (60) and Corollary 1. □

Proof of Proposition 3
For all $i$, one can show that $\lim_{z_t \to -\infty} c^\ast_{it} = \lim_{z_t \to -\infty} (1 - \tau_t)^{\frac{i}{n} - 1} \lambda_t^{\frac{i}{n}} e^\frac{z_t}{\lambda_t} = 0$, and

$$
\lim_{z_t \to +\infty} \frac{\partial c^\ast_{it}}{\partial W_t} = \lim_{z_t \to +\infty} (1 - \tau_t)^{\frac{i}{n} - 1} \left[ a_i + \sum_{j=1}^{i-1} \frac{a_j \gamma_i}{\gamma_j} \frac{1}{\lambda_t^{\frac{i}{n}}} e^\frac{z_t}{\lambda_t} \left( \frac{1}{\gamma_j} - \frac{1}{\gamma_i} \right) + \sum_{j=i+1}^{n} \frac{a_j \gamma_i}{\gamma_j} \frac{1}{\lambda_t^{\frac{i}{n}}} e^\frac{z_t}{\lambda_t} \left( \frac{1}{\gamma_j} - \frac{1}{\gamma_i} \right) \right]^{-1}
$$

If $\gamma_j > (\langle) \gamma_i$, $\lim_{z_t \to +\infty} e^\left( \frac{1}{\gamma_j} - \frac{1}{\gamma_i} \right) z_t = +\infty(0)$ and $\lim_{z_t \to -\infty} e^\left( \frac{1}{\gamma_j} - \frac{1}{\gamma_i} \right) z_t = 0(+\infty)$. Thus

$$
\lim_{z_t \to +\infty} \frac{\partial c^\ast_{it}}{\partial W_t} = \frac{1}{a_i} \quad \text{and} \quad \lim_{z_t \to +\infty} \frac{\partial^2 c^\ast_{it}}{\partial W_t^2} = 0 \quad \text{for} \quad i = 2, 3, ..., n \quad (61)
$$

The second derivative of the payout policy with respect to total net worth is

$$
\frac{\partial^2 c^\ast_{it}}{\partial W_t^2} = \frac{(1 - \tau_t)^{\frac{i}{n} - 1} (\lambda_t e^{\gamma_t}) \frac{i}{n} \sum_{j=1}^{n} \frac{a_j}{\gamma_j} (\lambda_j e^{\gamma_t}) \left( \frac{1}{\gamma_i} - \frac{1}{\gamma_j} \right)}{\gamma_i W'(z_t)^3} > (\langle) \quad (62)
$$

$$
\equiv \sum_{j=1}^{i-1} \frac{a_j}{\gamma_j} (\lambda_j e^{\gamma_t}) \left( \frac{1}{\gamma_j} - \frac{1}{\gamma_i} \right) < (\rangle) \sum_{j=i+1}^{n} \frac{a_j}{\gamma_j} (\lambda_j e^{\gamma_t}) \left( \frac{1}{\gamma_i} - \frac{1}{\gamma_j} \right)
$$

For $i = 1$ and $i = n$, we can easily see that $\frac{\partial^2 c^\ast_{it}}{\partial W_t^2} > 0$ and $\frac{\partial^2 c^\ast_{it}}{\partial W_t^2} < 0$. Both sides of (62) are positive and convex monotonically increasing functions of $z_t$. Moreover, because the exponential power is such that $\frac{1}{\gamma_n} < ... < \frac{1}{\gamma_{i+1}} < \frac{1}{\gamma_i} < ... < \frac{1}{\gamma_1}$, we can see that when $z_t \to -\infty$, the left-hand side of (62) is dominated by the right-hand side, while $z_t \to +\infty$, the reverse holds. Thus, the single-crossing property of two convex monotonically increasing functions implies that for $i = 2, ..., n - 1$, there exist a $z_t^\ast \in \mathbb{R}$, e.g. $W_t^\ast = W(z_t^\ast)$, s.t

$$
\sum_{j=1}^{i-1} \frac{a_j}{\gamma_j} (\lambda_j e^{\gamma_t}) \left( \frac{1}{\gamma_j} - \frac{1}{\gamma_i} \right) = \sum_{j=i+1}^{n} \frac{a_j}{\gamma_j} (\lambda_j e^{\gamma_t}) \left( \frac{1}{\gamma_i} - \frac{1}{\gamma_j} \right)
$$

$$
\frac{\partial^2 c^\ast_{it}}{\partial W_t^2} \geq 0 \quad \text{for} \quad W_t \leq W(z_t^\ast) \quad \text{and} \quad \frac{\partial^2 c^\ast_{it}}{\partial W_t^2} \leq 0 \quad \text{for} \quad W_t \geq W(z_t^\ast) \quad (64)
$$

Additionally, $W(z_2^\ast) \geq ... \geq W(z_{n-1}^\ast)$. In other words, the payout functions for intermediate risk averse investors is initially convex and then concave as the total net worth increases, and the inflexion point decreases as the degree of risk aversion increases. Lastly,
by setting $\gamma_1 = ... = \gamma_n = \gamma$, one can easily see that $c_{it}^*$ is linear in $W_t$. □

Proof of Proposition 4

Suppose that $\gamma_1 = ... = \gamma_n = \gamma$. We arrange $W_{i0}$ such that $W_{10} \geq W_{20} \geq ... \geq W_{n0}$. We normalize the utility weights such that they add up to 1. With $\lambda_i = \frac{W_{i0}^\gamma}{\sum_{j=1}^n W_{j0}^\gamma}$, the level of concentration measured by the Herfindahl index can then be written as

$$H = \sum_{i=1}^n \lambda_i^2 = \frac{\sum_{i=1}^n (W_{i0}^\gamma)^2}{\left[\sum_{j=1}^n W_{j0}^\gamma\right]^2} \quad (65)$$

Therefore, $H = \frac{1}{n}$ when $\gamma = 1$. $0 < H < 1$ follows immediately from the Cauchy–Schwarz inequality. Because $W_{i0} \geq W_{j0}$ for all $j \geq i$, we can further show that

$$\frac{\partial H}{\partial \gamma} = \frac{2}{\left(\sum_{j=1}^n W_{j0}^\gamma\right)^3} \left[\sum_{i=1}^n \sum_{j>i} W_{i0}^\gamma W_{j0}^\gamma (W_{i0}^\gamma - W_{j0}^\gamma) \ln \left(\frac{W_{i0}^\gamma}{W_{j0}^\gamma}\right)\right] \geq 0 \quad (66)$$

□

Proof of Proposition 5

The first equality follows directly from Equation (15). We prove the second equality by raising both sides of Equation (31) by the power of $\gamma_i$, which gives

$$\left(\frac{W_{i0}^s}{W_{i0}}\right)^{\gamma_i} = e^{\gamma_i \left[\frac{\alpha_i - \sigma_i^2}{2}t + \sigma_i B_t\right]} \quad \text{for} \quad i = 1, ..., n$$

Further simplification shows that the exponential power is indeed independent of risk aversion: $\gamma_i \left[\frac{(\alpha_i - \sigma_i^2)}{2}t + \sigma_i B_t\right] = -\left(\rho - \frac{1}{2}\xi^2 - r\right)t + \xi B_t$, implying

$$\left(\frac{W_{i0}^s}{a_i}\right)^{\gamma_i} = \frac{W_{i0}^s}{a_i} e^{\gamma_i \left[\frac{(\alpha_i - \sigma_i^2)}{2}t + \sigma_i B_t\right]} = \frac{W_{i0}^s}{a_i} e^{\gamma_i \left[\frac{(\alpha_i - \sigma_i^2)}{2}t + \sigma_i B_t\right]} = \frac{W_{i0}^s}{a_i} e^{\gamma_i \left[\frac{(\alpha_i - \sigma_i^2)}{2}t + \sigma_i B_t\right]} = \frac{\lambda_i}{\lambda_j} \quad (67)$$

□
Proof of Proposition 6

Since $c^*_{it} = (1 - \tau_p)\frac{1}{\gamma_i} (\lambda_i e^{z_i})^{\frac{1}{\gamma_i}} = \frac{W_{it}}{a_i}$ and $c^*_{it} = \frac{W^*_t}{a_i}$, it is enough to show that $W_{it}$ and $W^*_{it}$ are two identical processes. By substituting the optimal investment and payout policies (Equations (9) and (10)) into Equation (4), one can get

$$dW_t = \left[ \xi^2 W'(z_t) + rW(z_t) - \sum_{i=1}^{n} (1 - \tau_p)\frac{1}{\gamma_i} (\lambda_i e^{z_i})^{\frac{1}{\gamma_i}} \right] dt + \xi W'(z_t)dB_t$$

Applying Itô’s Lemma to $z(W_t)$ gives

$$dz_t = \left[ \frac{\xi^2 W'(z_t)}{W(z_t)} + \frac{rW(z_t)}{W(z_t)} - \sum_{i=1}^{n} c^*_{it} - \frac{1}{2} \frac{\xi^2 W''(z_t)}{W'(z_t)} \right] dt + \xi dB_t \equiv \mu_z dt + \sigma_z dB_t$$

In the sole proprietorship, the net worth process for each investor, $W^*_t$, evolves according to Equation (30). We would like to compare the two claim value processes $W_{it}$ and $W^*_{it}$ to see whether joining a partnership with the weight $\lambda_i = \left( \frac{W_{i0}}{a_i} \right)^{\gamma_i} e^{-z_0}$ can create value for individuals. Applying Itô’s lemma to $W_{it} = a_i \lambda_i^{\frac{1}{\gamma_i}} e^{z_i}$ shows that

$$\frac{dW_{it}}{W_{it}} = \left[ \frac{\mu_z}{\gamma_i} + \frac{1}{2} \frac{\sigma^2_z}{\gamma_i^2} \right] dt + \frac{\sigma_z}{\gamma_i} dB_t \equiv \alpha_{W_t} dt + \sigma_{W_t} dB_t$$

All we need to check is the following: $\alpha_{W_{it}} = \alpha_i$ and $\sigma_{W_{it}} = \sigma_i$, that is, $\frac{\mu_z}{\gamma_i} + \frac{1}{2} \frac{\sigma^2_z}{\gamma_i^2} = \frac{\xi^2}{\gamma_i} + r - \frac{1}{a_i}$ and $\frac{\sigma_z}{\gamma_i} = \frac{\xi}{\gamma_i}$. The second equality is satisfied as $\sigma_z = \xi$. The first equality also holds as

$$\alpha_{W_{it}} - \alpha_i = \frac{\xi^2}{\gamma_i} + \frac{rW(z_t)}{\gamma_i W'(z_t)} - \sum_{i=1}^{n} c^*_{it} - \frac{1}{2} \frac{\xi^2 W''(z_t)}{\gamma_i W'(z_t)} - \sum_{i=1}^{n} \left( r - \frac{\xi^2}{2\gamma_i} - 2(\rho + \frac{1}{2}\xi^2 - r)\gamma_i - 2r\gamma_i^2 \right) = 0$$

Given that the initial values are the same ($W_{i0} = W^*_{i0}$), we can conclude that $W_{it}$ and $W^*_{it}$ are indeed two identical processes. This means $W_{it} = W^*_{it}$ for all $t$ and $i = 1, ..., n$. Similarly, we conclude that $c^*_{it}$ and $c^*_{it}$ are also two identical processes given $c^*_{i0} = c^*_{i0}$. □
Proof of Proposition 7

We normalize the utility weight such that they add up to 1 and differentiate \( \lambda_i \) w.r.t \( \tau_p \):

\[
\frac{\partial \lambda_i}{\partial \tau_p} = \frac{\kappa_i (1 - \tau_p)^{-1}}{\sum_{j=1}^{n} \left( \frac{W_{ij}}{a_j} \right)^{\gamma_j}} \left[ \sum_{j=1}^{n} \left( \frac{W_{ij}}{a_j} \right)^{\gamma_j} (\gamma_j - \gamma_i) \right] = \frac{\kappa_i (1 - \tau_p)^{-1}}{\sum_{j=1}^{n} \left( \frac{W_{ij}}{a_j} \right)^{\gamma_j}} f_{\tau_p}(\gamma_i) 
\]

(70)

Therefore, \( \text{sign} \left( \frac{\partial \lambda_i}{\partial \tau_p} \right) = \text{sign} \left( f_{\tau_p}(\gamma_i) \right) \). One can show that \( f'_{\tau_p}(\gamma_i) < 0 \), meaning \( f_{\tau_p}(\gamma_i) \) is monotonically decreasing in \( \gamma_i \). We can see that \( f_{\tau_p}(\gamma_1) > 0 \) and \( f_{\tau_p}(\gamma_n) < 0 \). Therefore, \( f_{\tau_p}(\gamma_i) \) is a function such that it is initially positive and then turns negative as \( \gamma_i \) increases from \( \gamma_1 \) to \( \gamma_n \), i.e. \( \exists \gamma \in (\gamma_1, \gamma_n) \) such that

\[
f_{\tau_p}(\gamma) = \sum_{j=1}^{n} \left( \frac{W_{ij}}{a_j} \right)^{\gamma_j} (\gamma_j - \gamma_i) = 0 
\]

(71)

\[
\frac{\partial \lambda_i}{\partial \tau_p} \geq 0 \quad \text{for } \gamma_1 \leq \gamma_i \leq \gamma \quad \text{and} \quad \frac{\partial \lambda_i}{\partial \tau_p} \leq 0 \quad \text{for } \gamma \leq \gamma_i \leq \gamma_n 
\]

(72)

Taking the derivative of \( \lambda_i \) with respect to \( \tau_c \) gives

\[
\frac{\partial \lambda_i}{\partial \tau_c} = \frac{r' \left( \frac{W_{ai}}{a_i} \right)^{\gamma_i}}{\left( \sum_{j=1}^{n} \left( \frac{W_{ij}}{a_j} \right)^{\gamma_j} \right)^2} \left[ \sum_{j=1}^{n} \left( \frac{W_{ij}}{a_j} \right)^{\gamma_j} \left( \gamma_j (\gamma_j - 1) - \gamma_i (\gamma_i - 1) \right) \right] = \frac{r' \left( \frac{W_{AI}}{A_i} \right)^{\gamma_i}}{\left( \sum_{j=1}^{n} \left( \frac{W_{ij}}{a_j} \right)^{\gamma_j} \right)^2} f_{\tau_c}(\gamma_i) 
\]

Therefore, \( \text{sign} \left( \frac{\partial \lambda_i}{\partial \tau_c} \right) = \text{sign} \left( f_{\tau_c}(\gamma_i) \right) \). One can prove that \( \frac{d}{d\gamma_i} (\gamma_i(1 - \gamma_i)) \cdot \frac{D_i}{D_i} > 0 \), which implies \( f_{\tau_c}(\gamma_1) > 0 \) and \( f_{\tau_c}(\gamma_n) < 0 \). We can further prove that \( f'_{\tau_c}(\gamma_i) < 0 \). This means that \( f_{\tau_c}(\gamma_i) \) is a function such that it is initially positive and then turn negative as \( \gamma_i \) increases from \( \gamma_1 \) to \( \gamma_n \). Therefore, there exists an \( \tau \in (\gamma_1, \gamma_n) \) such that

\[
f_{\tau_c}(\gamma) = \sum_{j=1}^{n} \left( \frac{W_{ij}}{a_j} \right)^{\gamma_j} \left( \gamma_j (\gamma_j - 1) - \frac{\tau (1 - \tau)}{\tau} \right) = 0 
\]

(73)

\[
\frac{\partial \lambda_i}{\partial \tau_c} \geq 0 \quad \text{for } \gamma_1 \leq \gamma_i \leq \tau \quad \text{and} \quad \frac{\partial \lambda_i}{\partial \tau_c} \leq 0 \quad \text{for } \tau \leq \gamma_i \leq \gamma_n 
\]

(74)

\( \blacksquare \)
Proof of Proposition 8

When the utility weights are fixed, for a given $W_t$, the partial derivative of $c^*_it$ w.r.t $\tau_p$ is

$$\frac{\partial c^*_it}{\partial \tau_p} = \frac{c^*_it}{1 - \tau_p} \left[ \left( 1 - \frac{1}{\gamma_i} \right) - \frac{1}{\gamma_i W'(z_t)} \sum_{j=1}^{n} \left( 1 - \frac{1}{\gamma_j} \right) a_j(\lambda_j e^{\hat{z}_i})^{\frac{1}{\gamma_j}} \right]$$

(75)

Hence, the sign of terms in the square bracket determine the sign of the derivative:

$$\frac{\partial c^*_it}{\partial \tau_p} > (<) 0 \iff \sum_{j=1}^{i-1} \left( \frac{\gamma_i}{\gamma_j} - 1 \right) a_j(\lambda_j e^{\hat{z}_i})^{\frac{1}{\gamma_j}} > (<) \sum_{j=i+1}^{n} \left( 1 - \frac{\gamma_i}{\gamma_j} \right) a_j(\lambda_j e^{\hat{z}_i})^{\frac{1}{\gamma_j}}$$

(76)

For $i = 1(n)$, $\frac{\gamma_i}{\gamma_j} - 1 \leq (\geq) 0$ for all $j = 1, ..., n$, which implies $\frac{\partial c^*_it}{\partial \tau_p} < 0$ and $\frac{\partial c^*_it}{\partial \tau_p} > 0$. Both sides of (76) are positive and convex monotonically increasing functions of $z_t$. When $z_t \to -\infty(+\infty)$, the left-hand side of the Equation (76) is smaller (greater) than the right-hand side. Thus, the single-crossing property implies that for $i = 2, ..., n - 1$, there exists $\hat{z}_i \in \mathbb{R}$ such that

$$\sum_{j=1}^{i-1} \left( \frac{\gamma_i}{\gamma_j} - 1 \right) a_j(\lambda_j e^{\hat{z}_i})^{\frac{1}{\gamma_j}} = \sum_{j=i+1}^{n} \left( 1 - \frac{\gamma_i}{\gamma_j} \right) a_j(\lambda_j e^{\hat{z}_i})^{\frac{1}{\gamma_j}}$$

(77)

$$\frac{\partial c^*_it}{\partial \tau_p} \leq 0 \text{ for } W_t \leq W(\hat{z}_i) \text{ and } \frac{\partial c^*_it}{\partial \tau_p} \geq 0 \text{ for } W_t \geq W(\hat{z}_i)$$

(78)

In addition, the cutoff boundary $\hat{z}_i$ decreases in $\gamma_i$, that is, $W(\hat{z}_2) \geq ..., \geq W(\hat{z}_{n-1})$. For the claim values, one can show that $\frac{\partial \hat{z}_i}{\partial \tau_p} = \frac{\partial c^*_it}{\partial \tau_p} \cdot \frac{W_{it}}{c^*_it}$. The results for $W_{it}$ follow naturally.

□

Proof of Proposition 9
When the utility weights are fixed, the partial derivative of $W_{it}$ w.r.t $\tau_c$ for a given $W_t$ is

$$\frac{\partial W_{it}}{\partial \tau_c} = r'W_{it} \left[ \frac{\gamma_i(1 - \gamma_i)}{D_i} - \frac{1}{\gamma_i W'(z_i)} \sum_{j=1}^n r'\gamma_j(1 - \gamma_j) a_j(\lambda_j e^{z_j})^{\frac{1}{\gamma_j}} \right] > (\leq) 0 \quad (79)$$

$$\Leftrightarrow \sum_{j=1}^n a_j(\lambda_j e^{z_j})^{\frac{1}{\gamma_j}} \left[ \frac{\gamma_i(\gamma_i - 1)}{\gamma_j D_i} - \frac{\gamma_j(\gamma_j - 1)}{\gamma_i D_j} \right] > (\leq) 0 \quad (80)$$

Since $\frac{d}{d\gamma_i} \left( \frac{\gamma_i(\gamma_i - 1)}{D_i} \right) > 0$, for $i = 1$ or $n$, we get $\frac{r'\gamma_j(1 - \gamma_j) a_j(\lambda_j e^{z_j})^{\frac{1}{\gamma_j}}}{\gamma_i W'(z_i)} < \frac{\gamma_i(\gamma_i - 1)}{\gamma_j D_j}$ and $\frac{\gamma_i(\gamma_i - 1)}{\gamma_i D_i} > \frac{\gamma_i(\gamma_i - 1)}{\gamma_j D_j}$ for all $j = 1, \ldots, n$. As a result, $\frac{\partial W_{it}}{\partial \tau_c} < 0$ and $\frac{\partial W_{it}}{\partial \tau_c} > 0$. For $i = 2, \ldots, n$, applying the method used in the proof of Proposition 8, one can show that there exists $\bar{z}_{iw} \in \mathbb{R}$ s.t

$$\sum_{j=1}^n a_j(\lambda_j e^{z_j})^{\frac{1}{\gamma_j}} \left[ \frac{\gamma_i(\gamma_i - 1)}{\gamma_j D_i} - \frac{\gamma_j(\gamma_j - 1)}{\gamma_i D_j} \right] = 0 \quad (81)$$

$$\frac{\partial W_{it}}{\partial \tau_c} \leq 0 \text{ for } W_t \leq W(\bar{z}_{iw}) \text{ and } \frac{\partial W_{it}}{\partial \tau_c} \geq 0 \text{ for } W_t \geq W(\bar{z}_{iw}) \quad (82)$$

For the payout, the partial derivative of $c^*_{it}$ w.r.t $\tau_c$ for a given $W_t$ is

$$\frac{\partial c^*_{it}}{\partial \tau_c} = c^*_{it} \frac{dz_i}{d\tau_c} = c^*_{it} \frac{1}{\gamma_i W'(z_i)} \sum_{j=1}^n (1 - \gamma_j) r'\gamma_j a_j(\lambda_j e^{z_j})^{\frac{1}{\gamma_j}} > (\leq) 0 \quad (83)$$

$$\Leftrightarrow \sum_{j=1}^k (1 - \gamma_j) r'\gamma_j a_j(\lambda_j e^{z_j})^{\frac{1}{\gamma_j}} > (\leq) \sum_{j=k+1}^n (\gamma_j - 1) r'\gamma_j a_j(\lambda_j e^{z_j})^{\frac{1}{\gamma_j}} \quad (84)$$

If for all $i = 1, \ldots, n$, $\gamma_i \geq (\leq)1$, Equation (83) implies $\frac{\partial c^*_{it}}{\partial \tau_c} \leq (\geq)0$. When $\gamma_1 \leq \ldots \leq \gamma_k \leq 1 \leq \gamma_{k+1} \leq \ldots \leq \gamma_n$, since both sides of (84) are positive and convex monotonically increasing functions of $z_i$, we apply the method used in the proof of Proposition 8 to show that there exists $\bar{z}_c \in \mathbb{R}$ such that

$$\sum_{j=1}^k (1 - \gamma_j) r'\gamma_j a_j(\lambda_j e^{z_c})^{\frac{1}{\gamma_j}} = \sum_{j=k+1}^n (\gamma_j - 1) r'\gamma_j a_j(\lambda_j e^{z_c})^{\frac{1}{\gamma_j}} \quad (85)$$

$$\forall i = 1, \ldots, n, \frac{\partial c^*_{it}}{\partial \tau_c} \leq 0 \text{ for } W_t \leq W(\bar{z}_c) \text{ and } \frac{\partial c^*_{it}}{\partial \tau_c} \geq 0 \text{ for } W_t \geq W(\bar{z}_c) \quad (86)$$

□
References


Figure 1: Financial policies and claim values

The parameter values for the graphs are: risk free rate ($r = 0.05$), subjective discount rate ($\rho = 0.1$), personal tax rate ($\tau_p = 0$), corporate tax rate ($\tau_c = 0$), drift ($\mu = 0.12$), volatility ($\sigma = 0.3$) and the corresponding Sharpe ratio ($\xi = 0.233$). In this figure, we consider a group of 4 investors with heterogeneous risk aversion ($\gamma_1 = 0.5$, $\gamma_2 = 2$, $\gamma_3 = 3$, and $\gamma_4 = 9$) and exogenous initial endowments ($W_{10} = W_{20} = W_{30} = W_{40} = 15$). Panels A, B, C, D, E and F plot, respectively, investors’ payouts, the firm’s Net Debt Ratio, the firm’s payout yield, investors’ certainty equivalent claim values, investors’ control weights and the group’s level of relative risk aversion as functions of the firm’s net worth $W_t$.

In panels B and C, the thick solid lines represent the NDR and payout yield based on group decision making, whereas the other lines represent an individual investor’s optimal policy.
Table 1: Comparative statics of investors’ utility weights and life-time utility

The table presents investors’ utility weights ($\lambda_i$) and life-time utility ($J_i$) for different levels of volatility ($\sigma$), risk-free rate ($r$), personal tax rate ($\tau_p$) and corporate tax rate ($\tau_c$). We consider a group of 10 investors ($i = 1, ..., 10$). Each investor has the same initial capital contribution ($W_{i0}$) but a different coefficient of relative risk aversion ($\gamma_i$). Each panel corresponds to a different level of initial capital contribution ($W_{i0} = 5, 10$ or $15$). The baseline parameter values are: $\mu = 0.12$, $\sigma = 0.3$, $r = 0.05$, $\tau_p = 0$ and $\tau_c = 0$. The other parameter values are: the subjective discount rate ($\rho$) = 0.1, and the drift ($\mu$) = 0.12.

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Panel A: Low level of initial capital contribution

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Panel B: Medium level of initial capital contribution

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Panel C: High level of initial capital contribution

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Panel C: High level of initial capital contribution

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